ANSWERS

1. Is there any other point on the globe, besides the North Pole, from which you could walk a mile south, a mile east, and a mile north and find yourself back at the starting point? Yes indeed; not just one point but an infinite number of them! You could start from any point on a circle drawn around the South Pole at a distance slightly more than $1 + 1/2\pi$ miles (about 1.16 miles) from the Pole—the distance is "slightly more" to take into account the curvature of the earth. After walking a mile south, your next walk of one mile east will take you on a complete circle around the Pole, and the walk one mile north from there will then return you to the starting point. Thus your starting point could be any one of the infinite number of points on the circle with a radius of about 1.16 miles from the South Pole. But this is not all. You could also start at points closer to the Pole, so that the walk east would carry you just twice around the Pole, or three times, and so on.

2. There are 88 winning first hands. They fall into two categories: (1) four tens and any other card (48 hands); (2) three tens and any of the following pairs from the suit not represented by a ten: A-9, K-9, Q-9, J-9, K-8, Q-8, J-8, Q-7, J-7, J-6 (40 hands).

3. It is impossible to cover the mutilated chessboard (with two opposite corner squares cut off) with 31 dominoes, and the proof is easy. The two diagonally opposite corners are of the same color. Therefore their removal leaves a board with two more squares of one color than of the other. Each domino covers two squares of opposite

color, since only opposite colors are adjacent. After you have covered 60 squares with 30 dominoes, you are left with two uncovered squares of the same color. These two cannot be adjacent, therefore they cannot be covered by the last domino.

Suppose two cells of *opposite* color are removed from the chess-board. Can you prove that no matter which two cells are removed, the board can always be covered with 31 dominoes? For a simple, elegant proof that this is always possible, see my book *Aha! Insight* (W. H. Freeman, 1978, page 19).

4. If we require that the question be answerable by "yes" or "no," there are several solutions, all exploiting the same basic gimmick. For example, the logician points to one of the roads and says to the native, "If I were to ask you if this road leads to the village, would you say 'yes'?" The native is forced to give the right answer, even if he is a liar! If the road does lead to the village, the liar would say "no" to the direct question, but as the question is put, he lies and says he would respond "yes." Thus the logician can be certain that the road does lead to the village, whether the respondent is a truth-teller or a liar. On the other hand, if the road actually does not go to the village, the liar is forced in the same way to reply "no" to the inquirer's question.

A similar question would be, "If I asked a member of the other tribe whether this road leads to the village, would he say 'yes'?" To avoid the cloudiness that results from a question within a question, perhaps this phrasing (suggested by Warren C. Haggstrom, of Ann Arbor, Michigan) is best: "Of the two statements, 'You are a liar' and 'This road leads to the village,' is one and only one of them true?" Here again, a "yes" answer indicates it is the road, a "no" answer that it isn't, regardless of whether the native lies or tells the truth.

Dennis Sciama, Cambridge University cosmologist, and John McCarthy of Hanover, New Hampshire, called my attention to a delightful additional twist on the problem. "Suppose," Mr. McCarthy wrote (in a letter published in *Scientific American*, April 1957), "the logician knows that 'pish' and 'tush' are the native words for 'yes' and 'no' but has forgotten which is which, though otherwise he can speak the native language. He can still determine which road leads to the village.

"He points to one of the roads and asks, 'If I asked you whether the road I am pointing to is the road to the village would you say pish?' If the native replies, 'Pish,' the logician can conclude that the road pointed to is the road to the village even though he will still be in the dark as to whether the native is a liar or a truth-teller and as to whether 'pish' means yes or no. If the native says, 'Tush,' he may draw the opposite conclusion."

For hundreds of ingenious problems involving truth-tellers and liars, see the puzzle books by mathematician-logician Raymond M. Smullyan.

5. You can learn the contents of all three boxes by drawing just one marble. The key to the solution is your knowledge that the labels on all three of the boxes are incorrect. You must draw a marble from the box labeled "black-white." Assume that the marble drawn is black. You know then that the other marble in this box must be black also, otherwise the label would be correct. Since you have now identified the box containing two black marbles, you can at once tell the contents of the box marked "white-white": you know it cannot contain two white marbles, because its label has to be wrong; it cannot contain two black marbles, for you have identified that box; therefore it must contain one black and one white marble. The third box, of course, must then be the one holding two white marbles. You can solve the puzzle by the same reasoning if the marble you draw from the "black-white" box happens to be white instead of black.

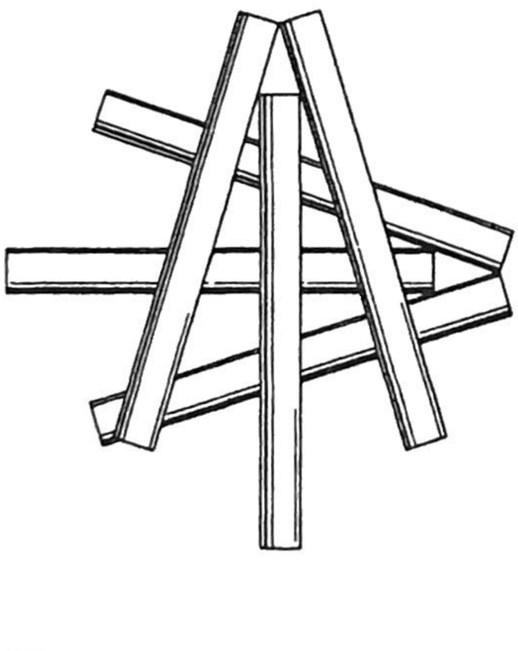
6. There is no way to reduce the cuts to fewer than six. This is at once apparent when you focus on the fact that a cube has six sides. The saw cuts straight—one side at a time. To cut the one-inch cube at the center (the one which has no exposed surfaces to start with) must take six passes of the saw.

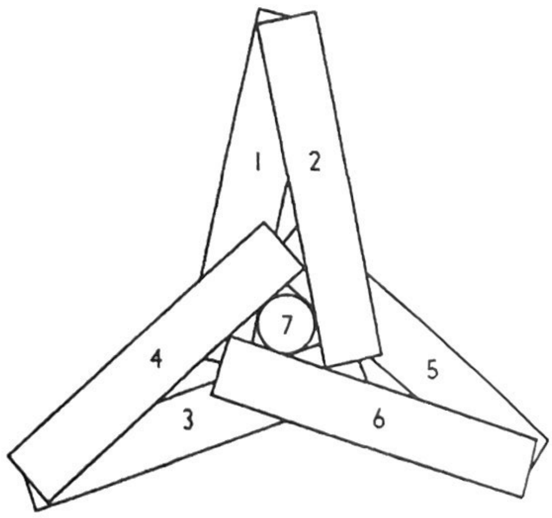
7. The answer to this puzzle is a simple matter of train schedules. While the Brooklyn and Bronx trains arrive equally often—at 10-minute intervals—it happens that their schedules are such that the Bronx train always comes to this platform one minute after the Brooklyn train. Thus the Bronx train will be the first to arrive only if the young man happens to come to the subway platform during this one-minute interval. If he enters the station at any other time—*i.e.*, during a nine-minute interval—the Brooklyn train will come first. Since the young man's arrival is random, the odds are nine to one for Brooklyn.

8. The commuter has walked for 55 minutes before his wife picks him up. Since they arrive home 10 minutes earlier than usual, this means that the wife has chopped 10 minutes from her usual travel time to and from the station, or five minutes from her travel time to the station. It follows that she met her husband five minutes before his usual pick-up time of five o'clock, or at 4:55. He started walking at four, therefore he walked for 55 minutes. The man's speed of walking, the wife's speed of driving and the distance between home and station are not needed for solving the problem. If you tried to solve it by juggling figures for these variables, you probably found the problem exasperating.

9. The counterfeit stack can be identified by a single weighing of coins. You take one coin from the first stack, two from the second, three from the third and so on to the entire 10 coins of the tenth stack. You then weigh the whole sample collection on the pointer scale. The excess weight of this collection, in number of grams, corresponds to the number of the counterfeit stack. For example, if the group of coins weighs seven grams more than it should, then the counterfeit stack must be the seventh one, from which you took seven coins (each weighing one gram more than a genuine half-dollar). Even if there had been an eleventh stack of ten coins, the procedure just described would still work, for no excess weight would indicate that the one remaining stack was counterfeit.

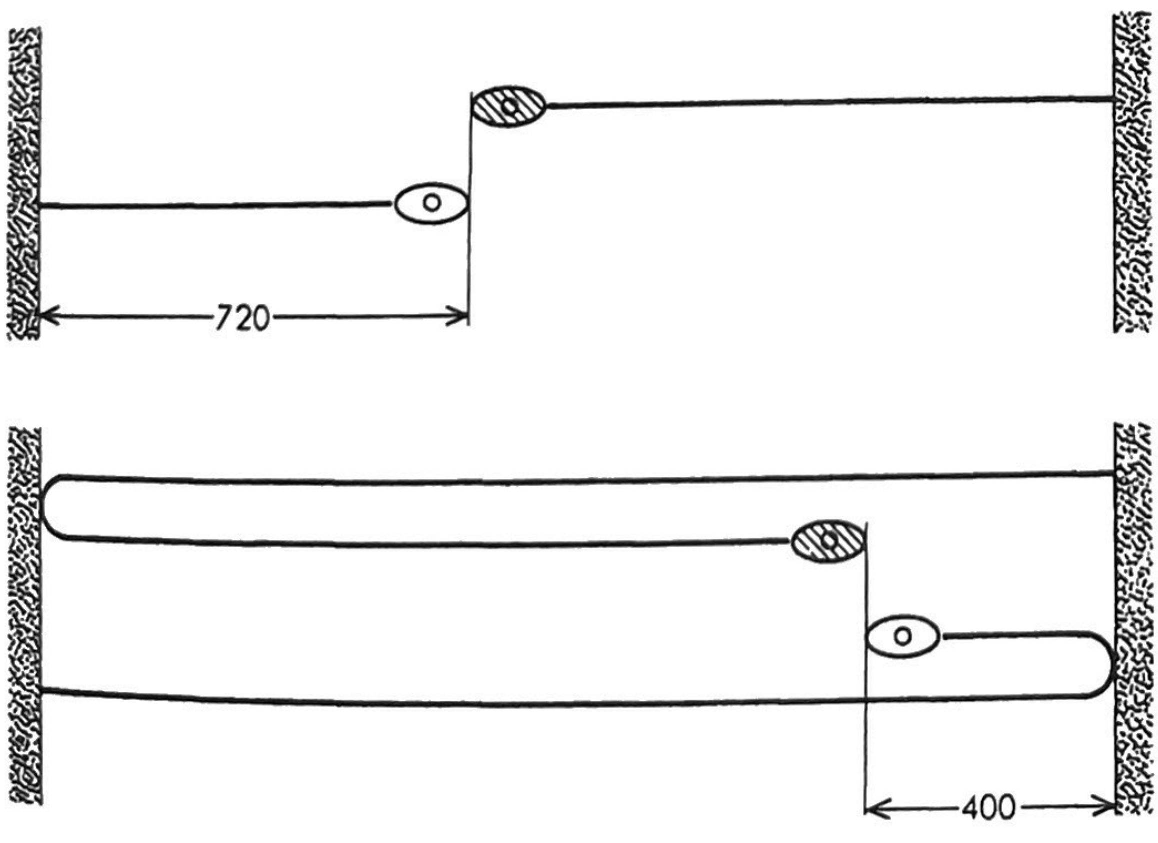
10. There are several different ways of placing the six cigarettes. The figure on the left shows the traditional solution as it is given in several old puzzle books.





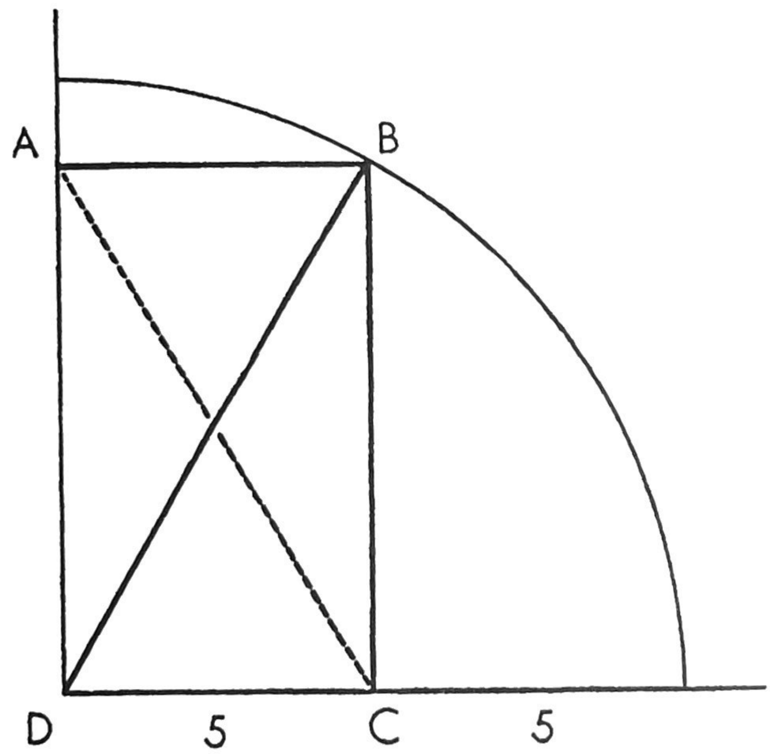
To my vast surprise, about fifteen readers discovered that *seven* cigarettes could also be placed so that each touched all of the others! This of course makes the older puzzle obsolete. The figure on the right, sent to me by George Rybicki and John Reynolds, graduate students in physics at Harvard, shows how it is done. "The diagram has been drawn," they write, "for the critical case where the ratio of length to diameter of the cigarettes is $\frac{7}{2}\sqrt{3}$. Here the points of contact occur right at the ends of the cigarettes. The solution obviously will work for any length-to-diameter ratio greater than $\frac{7}{2}\sqrt{3}$. Some observations on actual 'regular' size cigarettes give a ratio of about 8 to 1, which is, in fact, greater than $\frac{7}{2}\sqrt{3}$, so this is an acceptable solution." Note that if the center cigarette, pointing directly toward you in the diagram, is withdrawn, the remaining six provide a neat symmetrical solution of the original problem.

11. When the ferryboats meet for the first time [*top illustration*], the combined distance traveled by the boats is equal to the width of the river. When they reach the opposite shore, the combined distance is twice the width of the river; and when they meet the second time [*bottom figure*], the total distance is three times the river's width. Since the boats have been moving at a constant speed for the same period of time, it follows that each boat has gone three times as far as when they first met and had traveled a combined distance of one river-width. Since the white boat had traveled 720 yards when the first meeting occurred, its total distance at the time of the second meeting must be 3 × 720, or 2,160 yards. The bottom illustration shows clearly that this distance is 400 yards more than the river's width, so we subtract 400 from 2,160 to obtain 1,760 yards, or one mile, as the width of the river. The time the boats remained at their landings does not enter into the problem.

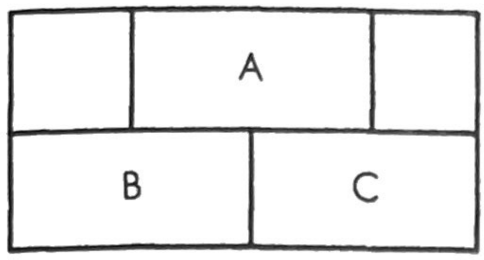


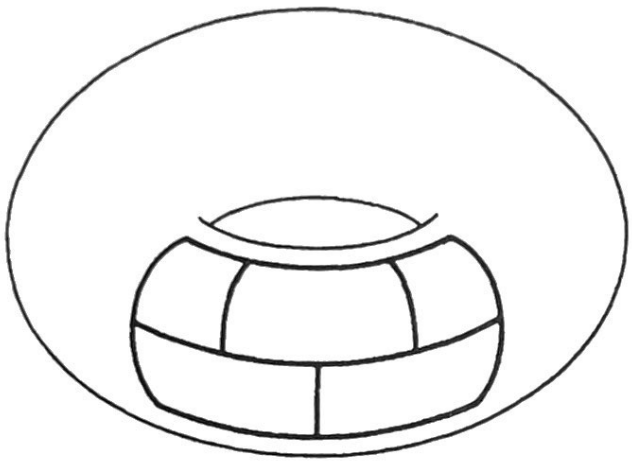
The problem can be approached in other ways. Many readers solved it as follows. Let x equal the river-width. On the first trip the ratio of distances traveled by the two boats is $x - 720{:}720$. On the second trip it is $2x - 400{:}x + 400$. These ratios are equal, so it is easy to solve for x.

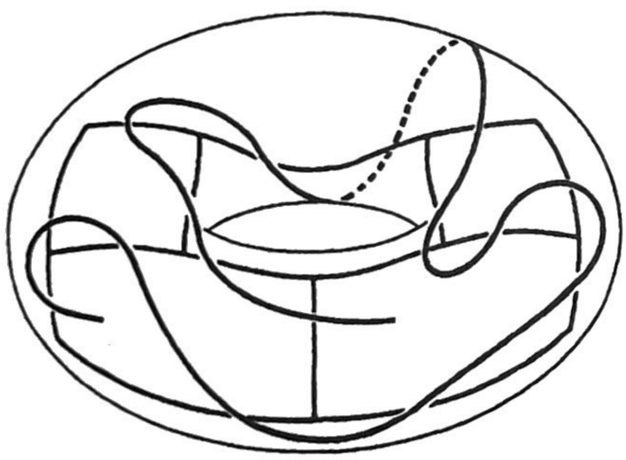
12. Line AC is one diagonal of the rectangle. The other diagonal is clearly the 10-unit radius of the circle. Since the diagonals are equal, line AC is 10 units long.



13. A continuous line that enters and leaves one of the rectangular spaces must of necessity cross two line segments. Since the spaces labeled A, B and C in the top illustration are each surrounded by an odd number of segments, it follows that an end of a line must be inside each if all segments of the network are crossed. But a continuous line has only two ends, so the puzzle is insoluble on a plane surface. This same reasoning applies if the network is on a sphere or on the side of a torus [*drawing at lower left*]. However, the network can be drawn on the torus [*drawing at lower right*] so that the hole of the torus is *inside* one of the three spaces, A, B and C. When this is done, the puzzle is easily solved.

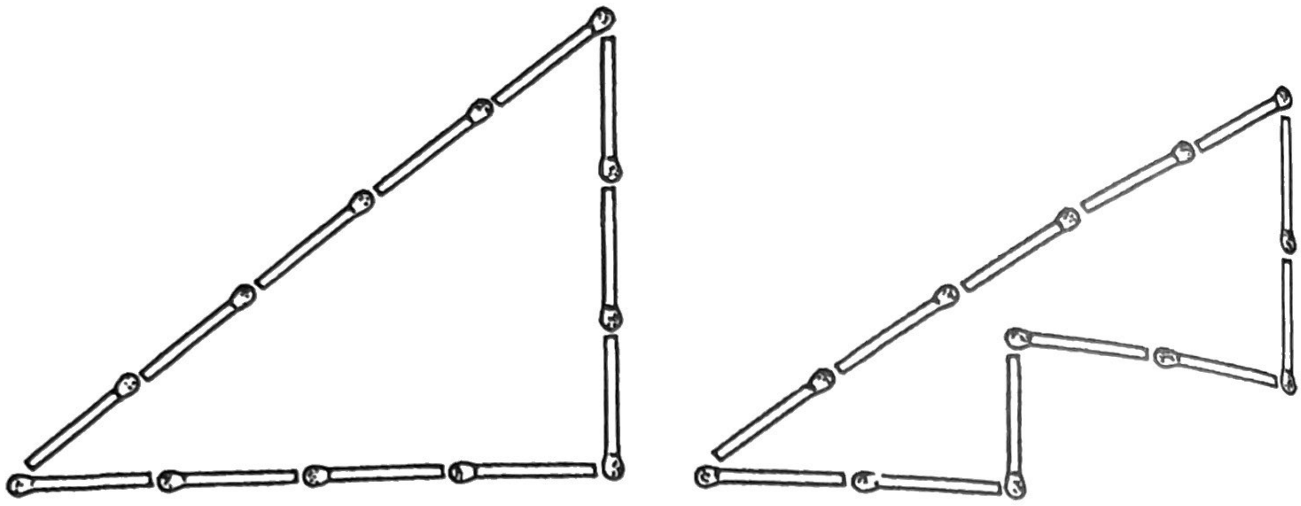




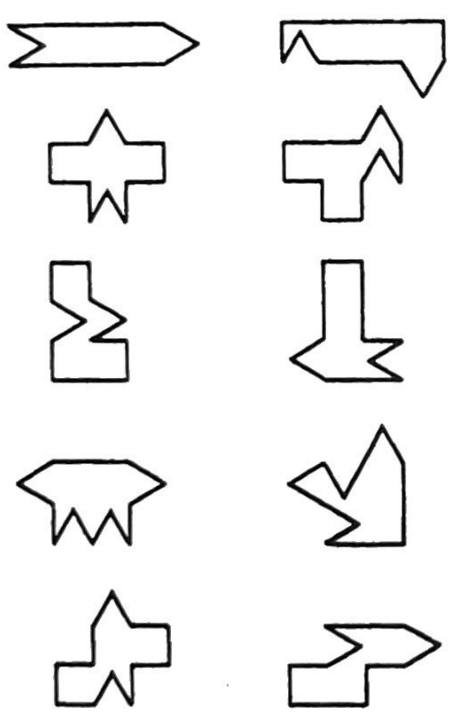


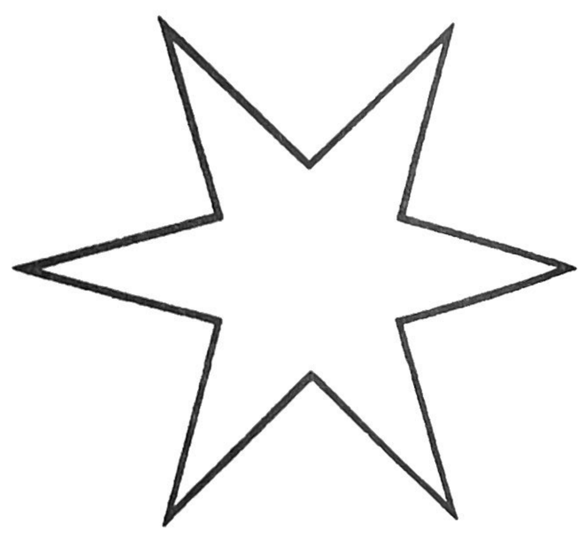
14. Twelve matches can be used to form a right triangle with sides of three, four and five units, as shown in the first illustration. This triangle will have an area of six square units. By altering the position of three matches as shown at right in the illustration, we remove two square units, leaving a polygon with an area of four.

This solution is the one to be found in many puzzle books. There are hundreds of other solutions. Elton M. Palmer, Oakmont, Pennsylvania, pointed out that each of the five tetrominoes (figures made with four squares) can provide the base for a large number of solutions. We simply add and subtract the same amount in triangular areas to accommodate all 12 matches. The second drawing depicts some representative samples, each row based on a different tetromino.



Eugene J. Putzer, staff scientist with the General Dynamics Corporation; Charles Shapiro, Oswego, New York; and Hugh J. Metz, Oak Ridge, Tennessee, suggested the star solution shown in the third drawing. By adjusting the width of the star's points you can produce any desired area between 0 and 11.196, the area of a regular dodecagon, the largest area possible with the 12 matches.



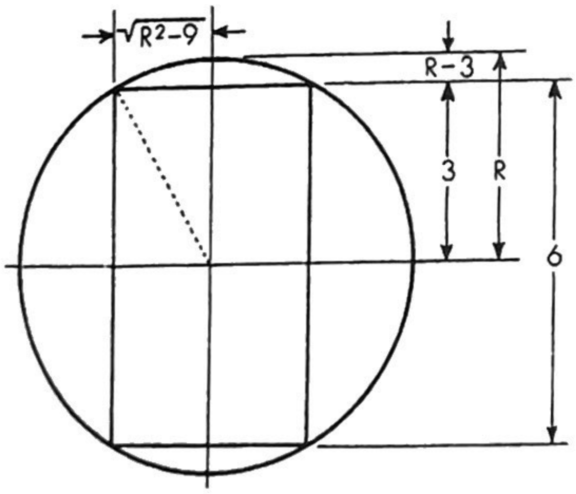


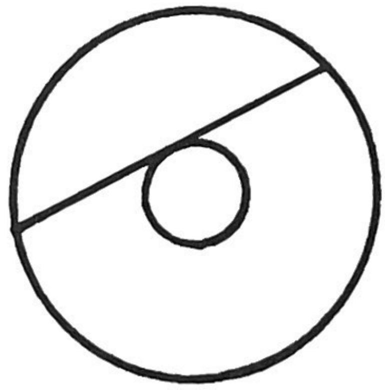
15. Without resorting to calculus, the problem can be solved as follows. Let R be the radius of the sphere. As the first illustration indicates, the radius of the cylindrical hole will then be the square root of $R^2 - 9$, and the altitude of the spherical caps at each end of the cylinder will be $R - 3$. To determine the residue after the cylinder and caps have been removed, we add the volume of the cylinder, $6\pi(R^2 - 9)$, to twice the volume of the spherical cap, and subtract the total from the volume of the sphere, $4\pi R^3/3$. The volume of the cap is obtained by the following formula, in which A stands for its altitude and r for its radius: $\pi A(3r^2 + A^2)/6$.

When this computation is made, all terms obligingly cancel out except 36π—the volume of the residue in cubic inches. In other words, the residue is constant regardless of the hole's diameter or the size of the sphere!

The earliest reference I have found for this beautiful problem is on page 86 of Samuel I. Jones's *Mathematical Nuts,* self-published, Nashville, 1932. A two-dimensional analog of the problem appears on page 93 of the same volume. Given the longest possible straight line that can be drawn on a circular track of any dimensions [*see second figure*], the area of the track will equal the area of a circle having the straight line as a diameter.

John W. Campbell, Jr., editor of *Astounding Science Fiction,* was one of several readers who solved the sphere problem quickly by reasoning adroitly as follows: The problem would not be given unless it has a unique solution. If it has a unique solution, the volume must be a constant which would hold even when the hole is reduced to zero radius. Therefore the residue must equal the volume of a sphere with a diameter of six inches, namely 36π.





16. At any given instant the four bugs form the corners of a square which shrinks and rotates as the bugs move closer together. The path of each pursuer will therefore at all times be perpendicular to the path of the pursued. This tells us that as A, for example, approaches B, there is no component in B's motion which carries B toward or away from A. Consequently A will capture B in the same time that it would take if B had remained stationary. The length of each spiral path will be the same as the side of the square: 10 inches.

If three bugs start from the corners of an equilateral triangle, each bug's motion will have a component of 1/2 (the cosine of a 60-degree angle is 1/2) its velocity that will carry it toward its pursuer. Two bugs will therefore have a mutual approach speed of 3/2 velocity. The bugs meet at the center of the triangle after a time interval equal to twice the side of the triangle divided by three times the velocity, each tracing a path that is 2/3 the length of the triangle's side.

For a generalization of this problem to n bugs at the corners of n-sided polygons see Chapter 24 of my *Sixth Book of Mathematical Games from Scientific American* (W. H. Freeman, 1971).

17. When Jones began to work on the professor's problem he knew that each of the four families had a different number of children, and that the total number was less than 18. He further knew that the product of the four numbers gave the professor's house number. Therefore his obvious first step was to factor the house number into four different numbers which together would total less than 18. If there had been only one way to do this, he would have immediately solved the problem. Since he could not solve it without further information, we conclude that there must have been more than one way of factoring the house number.

Our next step is to write down all possible combinations of four different numbers which total less than 18, and obtain the products of each group. We find that there are many cases where more than one combination gives the same product. How do we decide which product is the house number?

The clue lies in the fact that Jones asked if there was more than one child in the smallest family. This question is meaningful only if the house number is 120, which can be factored as $1 \times 3 \times 5 \times 8$, $1 \times 4 \times 5 \times 6$, or $2 \times 3 \times 4 \times 5$. Had Smith answered "No," the problem would remain unsolved. Since Jones did solve it, we know the answer was "Yes." The families therefore contained 2, 3, 4 and 5 children.

This problem was originated by Lester R. Ford and published in the *American Mathematical Monthly*, March 1948, as Problem E776.

18. The heads of the twiddled bolts move neither inward nor outward. The situation is comparable to that of a person walking up an escalator at the same rate that it is moving down.

19. Three airplanes are quite sufficient to ensure the flight of one plane around the world. There are many ways this can be done, but the following seems to be the most efficient. It uses only five tanks of fuel, allows the pilots of two planes sufficient time for a cup of coffee and a sandwich before refueling at the base, and there is a pleasing symmetry in the procedure.

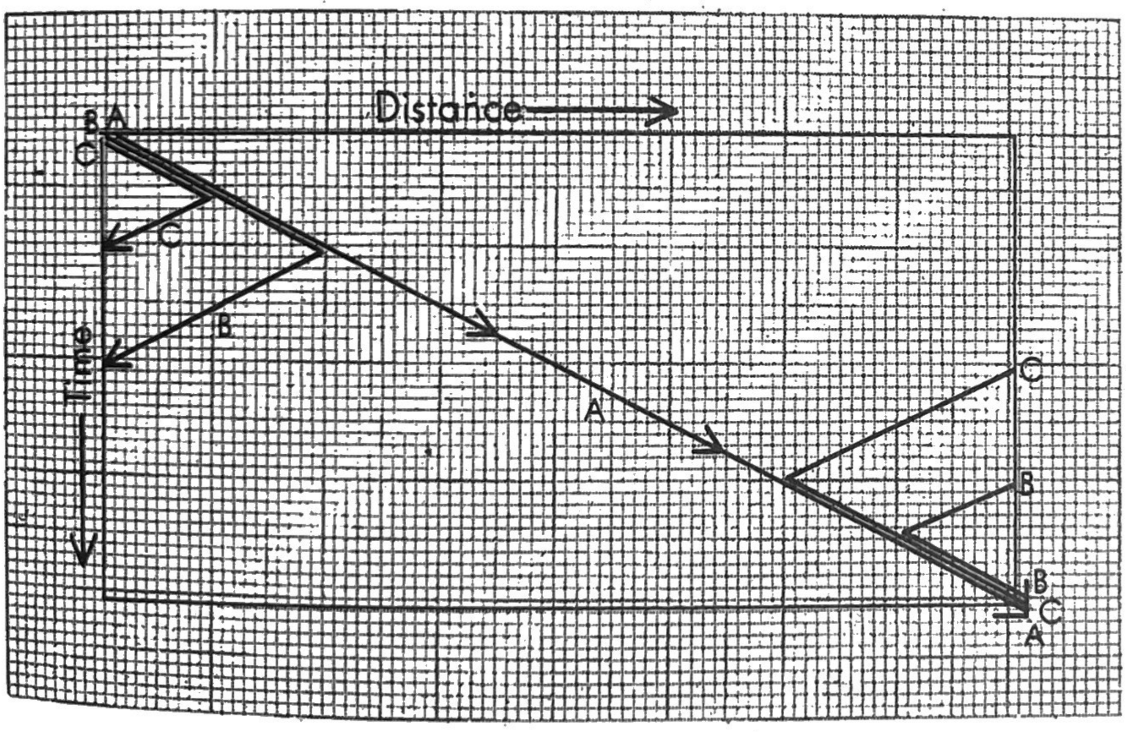
Planes A, B and C take off together. After going 1/8 of the distance, C transfers 1/4 tank to A and 1/4 to B. This leaves C with 1/4 tank, just enough to get back home.

Planes A and B continue another 1/8 of the way, then B transfers 1/4 tank to A. B now has 1/2 tank left, which is sufficient to get him back to the base where he arrives with an empty tank.

Plane A, with a full tank, continues until it runs out of fuel 1/4 of the way from the base. It is met by C which has been refueled at the base. C transfers 1/4 tank to A, and both planes head for home.

The two planes run out of fuel 1/8 of the way from the base, where they are met by refueled plane B. Plane B transfers 1/4 tank to each of the other two planes. The three planes now have just enough fuel to reach the base with empty tanks.

The entire procedure can be diagrammed as shown, where distance is the horizontal axis and time the vertical axis. The right and left edges of the chart should, of course, be regarded as joined.



20. Writing a three-digit number twice is the same as multiplying it by 1,001. This number has the factors 7, 11 and 13, so writing the chosen number twice is equivalent to multiplying it by 7, 11 and 13. Naturally when the product is successively divided by these same three numbers, the final remainder will be the original number.

I presented this classical trick before hand and desk calculators became omnipresent. The trick is most effective when a single spectator uses a calculator to perform his divisions while your back is turned, only to see his originally chosen number mysteriously appear in the readout after the final division.

21. The two missiles approach each other with combined speeds of 30,000 miles per hour, or 500 miles per minute. By running the scene backward in time we see that one minute before the collision the missiles would have to be 500 miles apart.

22. Number the top coin in the pyramid 1, the coins in the next row 2 and 3, and those in the bottom row 4, 5 and 6. The following four moves are typical of many possible solutions: Move 1 to touch 2 and 4, move 4 to touch 5 and 6, move 5 to touch 1 and 2 below, move 1 to touch 4 and 5.

23. Because two people are involved in every handshake, the total score for everyone at the convention will be evenly divisible by two and therefore even. The total score for the men who shook hands an even number of times is, of course, also even. If we subtract this even score from the even total score of the convention, we get an even total score for those men who shook hands an odd number of times. Only an even number of odd numbers will total an even number, so we conclude that an even number of men shook hands an odd number of times.

There are other ways to prove the theorem; one of the best was sent to me by Gerald K. Schoenfeld, a medical officer in the U.S. Navy. At the start of the convention, before any handshakes have occurred, the number of persons who have shaken hands an odd number of times will be 0. The first handshake produces two "odd persons." From now on, handshakes are of three types: between two even persons, two odd persons, or one odd and one even person. Each even-even shake increases the number of odd persons by 2. Each odd-odd shake decreases the number of odd persons by 2. Each odd-even shake changes an odd person to even and an even person to odd, leaving the number of odd persons unchanged. There is no way, therefore, that the even number of odd persons can shift its parity; it must remain at all times an even number.

Both proofs apply to a graph of dots on which lines are drawn to connect pairs of dots. The lines form a network on which the number of dots that mark the meeting of an odd number of lines is even.

24. In the triangular pistol duel the poorest shot, Jones, has the best chance to survive. Smith, who never misses, has the second best chance. Because Jones's two opponents will aim at each other when their turns come, Jones's best strategy is to fire into the air until one opponent is dead. He will then get the first shot at the survivor, which gives him a strong advantage.

Smith's survival chances are the easiest to determine. There is a chance of 1/2 that he will get the first shot in his duel with Brown, in which case he kills him. There is a chance of 1/2 that Brown will shoot first, and since Brown is 4/5 accurate, Smith has a 1/5 chance of surviving. So Smith's chance of surviving Brown is 1/2 added to 1/2 × 1/5 = 3/5. Jones, who is accurate half the time, now gets a crack at Smith. If he misses, Smith kills him, so Smith has a survival chance of 1/2 against Jones. Smith's over-all chance of surviving is therefore 3/5 × 1/2 = 3/10.

Brown's case is more complicated because we run into an infinite series of possibilities. His chance of surviving against Smith is 2/5 (we saw above that Smith's survival chance against Brown was 3/5, and since one of the two men must survive, we subtract 3/5 from 1 to obtain Brown's chance of surviving against Smith). Brown now faces fire from Jones. There is a chance of 1/2 that Jones will miss, in which case Brown has a 4/5 chance of killing Jones. Up to this point, then, his chance of killing Jones is 1/2 × 4/5 = 4/10. But there is a 1/5 chance that Brown may miss, giving Jones another shot. Brown's chance of surviving is 1/2; then he has a 4/5 chance of killing Jones again, so his chance of surviving on the second round is 1/2 × 1/5 × 1/2 × 4/5 = 4/100.

If Brown misses again, his chance of killing Jones on the third round will be 4/1,000; if he misses once more, his chance on the fourth round will be 4/10,000, and so on. Brown's total survival chance against Jones is thus the sum of the infinite series:

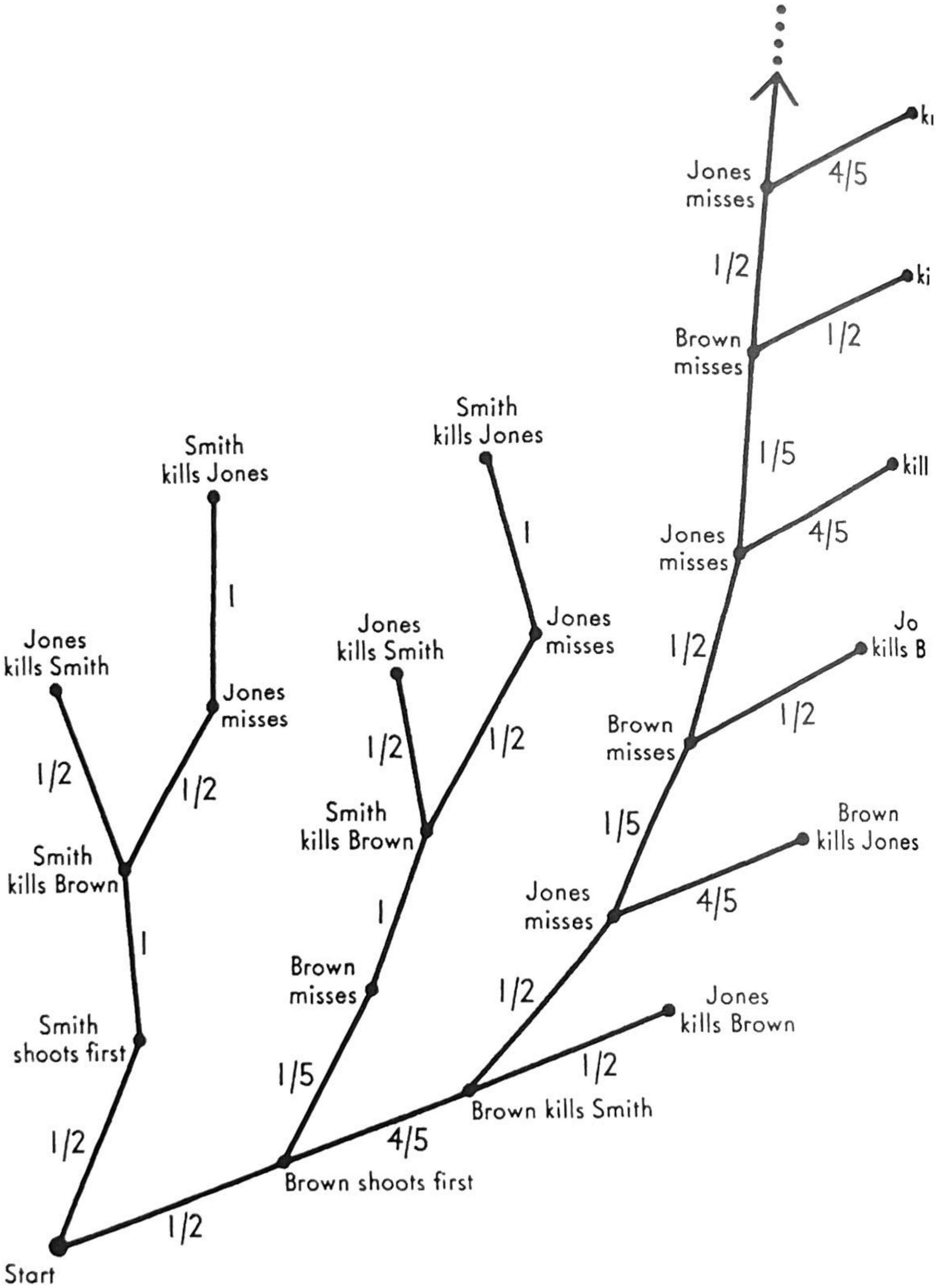
$$\frac{4}{10} + \frac{4}{100} + \frac{4}{1{,}000} + \frac{4}{10{,}000} \cdots$$

This can be written as the repeating decimal .444444 . . ., which is the decimal expansion of 4/9.

As we saw earlier, Brown had a 2/5 chance of surviving Smith; now we see that he has a 4/9 chance to survive Jones. His over-all survival chance is therefore 2/5 × 4/9 = 8/45.

Jones's survival chance can be determined in similar fashion, but of course we can get it immediately by subtracting Smith's chance, 3/10, and Brown's chance, 8/45, from 1. This gives Jones an over-all survival chance of 47/90.

The entire duel can be conveniently graphed by using the t diagram shown. It begins with only two branches because J passes if he has the first shot, leaving only two equal possibili · Smith shooting first or Brown shooting first, with intent to kill. branch of the tree goes on endlessly. The over-all survival chance an individual is computed as follows:



1. Mark all the ends of branches at which the person is survivor.

2. Trace each end back to the base of the tree, multiplying the probabilities of each segment as you go along. The product will be the probability of the event at the end of the branch.

3. Add together the probabilities of all the marked end-point events. The sum will be the over-all survival probability for that person.

In computing the survival chances of Brown and Jones, an infinite number of end-points are involved, but it is not difficult to see from the diagram how to formulate the infinite series that is involved in each case.

25. The following analysis of the desert-crossing problem appeared in an issue of *Eureka,* a publication of mathematics students at the University of Cambridge. Five hundred miles will be called a "unit"; gasoline sufficient to take the truck 500 miles will be called a "load"; and a "trip" is a journey of the truck in either direction from one stopping point to the next.

Two loads will carry the truck a maximum distance of 1 and 1/3 units. This is done in four trips by first setting up a cache at a spot 1/3 unit from the start. The truck begins with a full load, goes to the cache, leaves 1/3 load, returns, picks up another full load, arrives at the cache and picks up the cache's 1/3 load. It now has a full load, sufficient to take it the remaining distance to one unit.

Three loads will carry the truck 1 and 1/3 plus 1/5 units in a total of nine trips. The first cache is 1/5 unit from the start. Three trips put 6/5 loads in the cache. The truck returns, picks up the remaining full load and arrives at the first cache with 4/5 load in its tank. This, together with the fuel in the cache, makes two full loads, sufficient to carry the truck the remaining 1 and 1/3 units, as explained in the preceding paragraph.

We are asked for the minimum amount of fuel required to take the truck 800 miles. Three loads will take it 766 and 2/3 miles (1 and 1/3 plus 1/5 units), so we need a third cache at a distance of 33 and 1/3 miles (1/15 unit) from the start. In five trips the truck can build up this cache so that when the truck reaches the cache at the end of the seventh trip, the combined fuel of truck and cache will be three loads. As we have seen, this is sufficient to take the truck the remaining distance of 766 and 2/3 miles. Seven trips are made between starting point and first cache, using 7/15 load of gasoline. The three loads of fuel that remain are just sufficient for the rest of the way, so the total amount of gasoline consumed will be 3 and 7/15, or a little more than 3.46 loads. Sixteen trips are required.

Proceeding along similar lines, four loads will take the truck a distance of 1 and 1/3 plus 1/5 plus 1/7 units, with three caches located at the boundaries of these distances. The sum of this infinite series diverges as the number of loads increases; therefore the truck can cross a desert of any width. If the desert is 1,000 miles across, seven caches, 64 trips and 7.673 loads of gasoline are required.

Hundreds of letters were received on this problem, giving general solutions and interesting sidelights. Cecil G. Phipps, professor of mathematics at the University of Florida, summed matters up succinctly as follows:

"The general solution is given by the formula:

$$d = m\ (1 + 1/3 + 1/5 + 1/7 + \ldots),$$

where d is the distance to be traversed and m is the number of miles per load of gasoline. The number of depots to be established is one less than the number of terms in the series needed to exceed the value of d. One load of gasoline is used in the travel between each pair of stations. Since the series is divergent, any distance can be reached by this method although the amount of gasoline needed increases exponentially.

"If the truck is to return eventually to its home station, the formula becomes:

$$d = m\ (1/2 + 1/4 + 1/6 + 1/8 + \ldots)$$

This series is also divergent and the solution has properties similar to those for the one-way trip."

Many readers called attention to three previously published discussions of the problem:

"The Jeep Problem: A More General Solution." C. G. Phipps in the *American Mathematical Monthly,* Vol. 54, No. 8, pages 458–462, October 1947.

"Crossing the Desert." G. G. Alway in the *Mathematical Gazette,* Vol. 41, No. 337, page 209, October 1947.

Problem in Logistics: The Jeep Problem. Olaf Helmer. Project Rand Report No. RA-15015, December 1, 1946.

26. The key to Lord Dunsany's chess problem is the fact that the black queen is not on a black square as she must be at the start of a game. This means that the black king and queen have moved, and this could have happened only if some black pawns have moved. Pawns cannot move backward, so we are forced to conclude that the black pawns reached their present positions from the other side of the board! With this in mind, it is easy to discover that the white knight on the right has an easy mate in four moves.

White's first move is to jump his knight at the lower right corner of the board to the square just above his king. If black moves the upper left knight to the rook's file, white mates in two more moves. Black can, however, delay the mate one move by first moving his knight to the bishop's file instead of the rook's. White jumps his knight forward and right to the bishop's file, threatening mate on the next move. Black moves his knight forward to block the mate. White takes the knight with his queen, then mates with his knight on the fourth move.

27. In long division, when two digits are brought down instead of one, there must be a zero in the quotient. This occurs twice, so we know at once that the quotient is $x080x$. When the divisor is multiplied by the quotient's last digit, the product is a four-digit number. The quotient's last digit must therefore be 9, because eight times the divisor is a three-digit number.

The divisor must be less than 125 because eight times 125 is 1,000, a four-digit number. We now can deduce that the quotient's first digit must be more than 7, for seven times a divisor less than 125 would give a product that would leave more than two digits after it was subtracted from the first four digits in the dividend. This first digit cannot be 9 (which gives a four-digit number when the divisor is multiplied by it), so it must be 8, making the full quotient 80809.

The divisor must be more than 123 because 80809 times 123 is a seven-digit number and our dividend has eight digits. The only number between 123 and 125 is 124. We can now reconstruct the entire problem as follows:

```
        80809
124 | 10020316
       992
      ----
       1003
        992
       ----
        1116
        1116
        ----
```

28. Several procedures have been devised by which n persons can divide a cake in n pieces so that each is satisfied that he has at least $1/n$ of the cake. The following system has the merit of leaving no excess bits of cake.

Suppose there are five persons: A, B, C, D, E. A cuts off what he regards as 1/5 of the cake and what he is content to keep as his share. B now has the privilege, if he thinks A's slice is more than 1/5, of reducing it to what he thinks is 1/5 by cutting off a portion. Of course if he thinks it is 1/5 or less, he does not touch it. C, D and E in turn now have the same privilege. The last person to touch the slice keeps it as his share. Anyone who thinks that this person got less than 1/5 is naturally pleased because it means, in his eyes, that more than 4/5 remains. The remainder of the cake, including any cut-off pieces, is now divided among the remaining four persons in the same manner, then among three. The final division is made by one person cutting and the other choosing. The procedure is clearly applicable to any number of persons.

For a discussion of this and other solutions, see the section "Games of Fair Division," pages 363–368, in *Games and Decisions,* by R. Duncan Luce and Howard Raiffa, John Wiley and Sons, Inc., 1957.

The problem of dividing a cake between n persons so that each person is convinced he has his fair share has been the topic of many papers. Here are three:

"How to Cut a Cake Fairly," by L. E. Dubins and E. H. Spanier, in *American Mathematical Monthly,* Vol. 68, January 1961, pages 1–17.

"Preferred Shares," by Dominic Olivastro, in *The Sciences,* March/April 1992, pages 52–54.

"An Envy-Free Cake Division Algorithm," by Steven J. Brams and Alan D. Taylor, preprint. More than fifty references are cited in the bibliography.

29. The first sheet is folded as follows. Hold it face down so that when you look down on it the numbered squares are in this position:

2365
1874

Fold the right half on the left so that 5 goes on 2, 6 on 3, 4 on 1 and 7 on 8. Fold the bottom half up so that 4 goes on 5 and 7 on 6. Now tuck 4 and 5 between 6 and 3, and fold 1 and 2 under the packet.

The second sheet is first folded in half the long way, the numbers outside, and held so that 4536 is uppermost. Fold 4 on 5. The right end of the strip (squares 6 and 7) is pushed between 1 and 4, then bent around the folded edge of 4 so that 6 and 7 go between 8 and 5, and 3 and 2 go between 1 and 4.

For more puzzles involving paper folding see "The Combinatorics of Paper Folding," in my *Wheels, Life and Other Mathematical Amusements* (W. H. Freeman, 1983).

30. Regardless of how much wine is in one beaker and how much water is in the other, and regardless of how much liquid is transferred back and forth at each step (provided it is not all of the liquid in one beaker), it is impossible to reach a point at which the percentage of wine in each mixture is the same. This can be shown by a simple inductive argument. If beaker A contains a higher concentration of wine than beaker B, then a transfer from A to B will leave A with the higher concentration. Similarly a transfer from B to A—from a weaker to a stronger mixture—is sure to leave B weaker. Since every transfer is one of these two cases, it follows that beaker A must always contain a mixture with a higher percentage of wine than B. The only way to equalize the concentrations is by pouring all of one beaker into the other.

31. To determine the value of Brown's check, let x stand for the dollars and y for the cents. The problem can now be expressed by the following equation: $100y + x - 5 = 2(100x + y)$. This reduces to $98y - 199x = 5$, a Diophantine equation with an infinite number of integral solutions. A solution by the standard method of continued fractions gives as the lowest values in positive integers: $x = 31$ and $y = 63$, making Brown's check $31.63. This is a unique answer to the problem because the next lowest values are: $x = 129$, $y = 262$, which fails to meet the requirement that y be less than 100.

There is a much simpler approach to the problem and many readers wrote to tell me about it. As before, let x stand for the dollars on the check, y for the cents. After buying his newspaper, Brown has left $2x + 2y$. The change that he has left, from the x cents given him by the cashier, will be $x - 5$.

We know that y is less than 100, but we don't know yet whether it is less than 50 cents. If it is less than 50 cents, we can write the following equations:

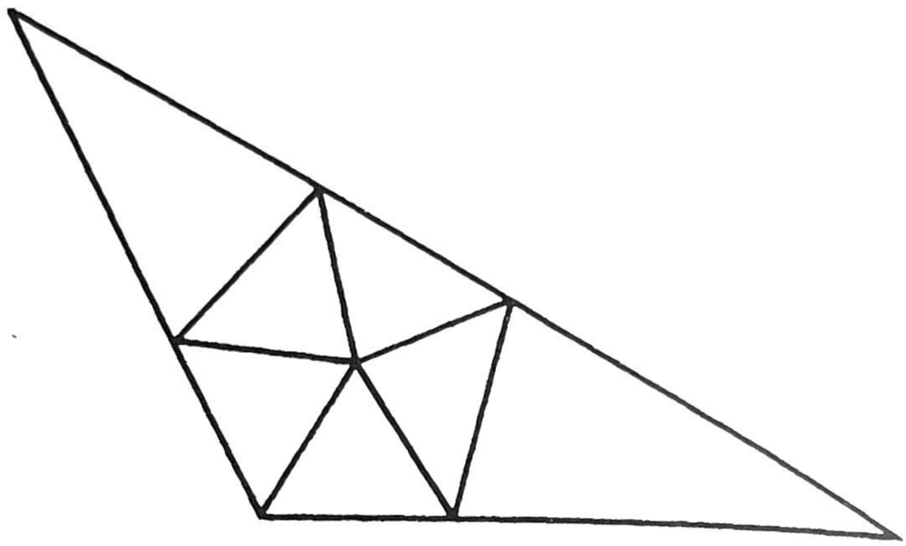
$$2x = y$$
$$2y = x - 5$$

If y is 50 cents or more, then Brown will be left with an amount of cents ($2y$) that is a dollar or more. We therefore have to modify the above equations by taking 100 from $2y$ and adding 1 to $2x$. The equations become:

$$2x + 1 = y$$
$$2y - 100 = x - 5$$

Each set of simultaneous equations is easily solved. The first set gives x a minus value, which is ruled out. The second set gives the correct values.

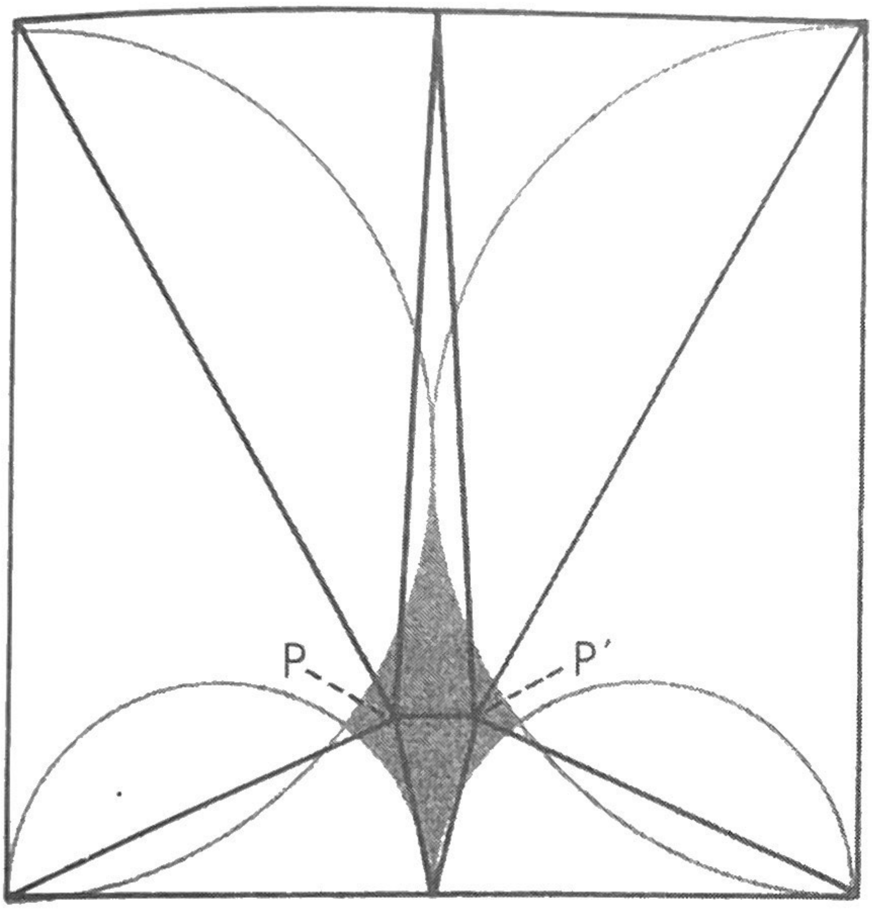
32. A number of readers sent "proofs" that an obtuse triangle cannot be dissected into acute triangles, but of course it can. The illustration shows a seven-piece pattern that applies to any obtuse triangle.



It is easy to see that seven is minimal. The obtuse angle must be divided by a line. This line cannot go all the way to the other side, for then it would form another obtuse triangle, which in turn would have to be dissected, consequently the pattern for the large triangle would not be minimal. The line dividing the obtuse angle must, therefore, terminate at a point *inside* the triangle. At this vertex, at least five lines must meet, otherwise the angles at this vertex would not all be acute. This creates the inner pentagon of five triangles, making a total of seven triangles in all. Wallace Manheimer, a Brooklyn high school teacher at the time, gave this proof as his solution to problem E1406 in *American Mathematical Monthly,* November 1960, page 923. He also showed how to construct the pattern for any obtuse triangle.

The question arises: Can any obtuse triangle be dissected into seven acute *isosceles* triangles? The answer is no. Verner E. Hoggatt, Jr., and Russ Denman (*American Mathematical Monthly,* November 1961, pages 912–913) proved that eight such triangles are sufficient for all obtuse triangles, and Free Jamison (*ibid.,* June–July 1962, pages 550–552) proved that eight are also necessary. These articles can be consulted for details as to conditions under which less than eight-piece patterns are possible. A right triangle and an acute nonisosceles triangle can each be cut into nine acute isosceles triangles, and an acute isosceles triangle can be cut into four congruent acute isosceles triangles similar to the original.

A square, I discovered, can be cut into eight acute triangles as shown. If the dissection has bilateral symmetry, points P and P′ must lie within the shaded area determined by the four semicircles. Donald L. Vanderpool pointed out in a letter that asymmetric distortions of the pattern are possible with point P anywhere outside the shaded area provided it remains outside the two large semicircles.



About 25 readers sent proofs, with varying degrees of formality, that the eight-piece dissection is minimal. One, by Harry Lindgren, appeared in *Australian Mathematics Teacher,* Vol. 18, pages 14–15, 1962. His proof also shows that the pattern, aside from the shifting of points P and P′ as noted above, is unique.

H. S. M. Coxeter pointed out the surprising fact that for any rectangle, even though its sides differ in length by an arbitrarily small amount, the line segment PP′ can be shifted to the center to give the pattern both horizontal and vertical symmetry.

Free Jamison found in 1968 that a square can be divided into ten acute isosceles triangles. See *The Fibonacci Quarterly* (December 1968) for a proof that a square can be dissected into any number of acute isosceles triangles equal or greater than 10.

33. The volume of a sphere is $4\pi/3$ times the cube of the radius. Its surface is 4π times the square of the radius. If we express the moon's radius in "lunars" and assume that its surface in square lunars equals its volume in cubic lunars, we can determine the length of the radius simply by equating the two formulas and solving for the value of the radius. Pi cancels out on both sides, and we find that the radius is three lunars. The moon's radius is 1,080 miles, so a lunar must be 360 miles.

34. Regardless of the number of slips involved in the game of Googol, the probability of picking the slip with the largest number (assuming that the best strategy is used) never drops below .367879. This is the reciprocal of *e,* and the limit of the probability of winning as the number of slips approaches infinity.

If there are ten slips (a convenient number to use in playing the game), the probability of picking the top number is .398. The strategy is to turn three slips, note the largest number among them, then pick the next slip that exceeds this number. In the long run you stand to win about two out of every five games.

What follows is a compressed account of a complete analysis of the game by Leo Moser and J. R. Pounder of the University of Alberta. Let n be the number of slips and p the number rejected before picking a number larger than any on the p slips. Number the slips serially from 1 to n. Let $k + 1$ be the number of the slip bearing the largest number. The top number will not be chosen unless k is equal to or greater than p (otherwise it will be rejected among the first p slips), and then only if the highest number from 1 to k is also the highest number from 1 to p (otherwise *this* number will be chosen before the top number is reached). The probability of finding the top number in case it is on the $k + 1$ slip is p/k, and the probability that the top number actually is on the $k + 1$ slip is $1/n$. Since the largest number can be on only one slip, we can write the following formula for the probability of finding it:

$$\frac{p}{n}\left(\frac{1}{p} + \frac{1}{p+1} + \frac{1}{p+2} \cdots\cdots + \frac{1}{n-1}\right)$$

Given a value for n (the number of slips) we can determine p (the number to reject) by picking a value for p that gives the greatest value to the above expression. As n approaches infinity, p/n approaches $1/e$, so a good estimate of p is simply the nearest integer to n/e. The general strategy, therefore, when the game is played with n slips, is to let n/e numbers go by, then pick the next number larger than the largest number on the n/e slips passed up.

This assumes, of course, that a player has no knowledge of the range of the numbers on the slips and therefore no basis for knowing whether a single number is high or low within the range. If one *has* such knowledge, the analysis does not apply. For example, if the game is played with the numbers on ten one-dollar bills, and your first draw is a bill with a number that begins with 9, your best strategy is to keep the bill. For similar reasons, the strategy in playing Googol is not strictly applicable to the unmarried girl problem, as many readers pointed out, because the girl presumably has a fair knowledge of the range in value of her suitors, and has certain standards in mind. If the first man who proposes comes very close to her ideal, wrote Joseph P. Robinson, "she would have rocks in her head if she did not accept at once."

Fox and Marnie apparently hit independently on a problem that had occurred to others a few years before. A number of readers said they had heard the problem before 1958—one recalled working on it in 1955—but I was unable to find any published reference to it. The problem of maximizing the *value* of the selected object (rather than the chance of getting the object of highest value) seems first to have been proposed by the famous mathematician Arthur Cayley in 1875. (See Leo Moser, "On a Problem of Cayley," in *Scripta Mathematica,* September–December 1956, pages 289–292.)

35. Let 1 be the length of the square of cadets and also the time it takes them to march this length. Their speed will also be 1. Let x be the total distance traveled by the dog and also its speed. On the dog's forward trip his speed relative to the cadets will be $x - 1$. On the return trip his speed relative to the cadets will be $x + 1$. Each trip is a distance of 1 (relative to the cadets), and the two trips are completed in unit time, so the following equation can be written:

$$\frac{1}{x-1} + \frac{1}{x+1} = 1$$

This can be expressed as the quadratic: $x^2 - 2x - 1 = 0$, for which x has the positive value of $1 + \sqrt{2}$. Multiply this by 50 to get the final answer: 120.7+ feet. In other words, the dog travels a total distance equal to the length of the square of cadets plus that same length times the square root of 2.

Loyd's version of the problem, in which the dog trots *around* the moving square, can be approached in exactly the same way. I paraphrase a clear, brief solution sent by Robert F. Jackson of the Computing Center at the University of Delaware.

As before, let 1 be the side of the square and also the time it takes the cadets to go 50 feet. Their speed will then also be 1. Let x be the distance traveled by the dog and also his speed. The dog's speed with respect to the speed of the square will be $x - 1$ on his forward trip, $\sqrt{x^2 - 1}$ on each of his two transverse trips, and $x + 1$ on his backward trip. The circuit is completed in unit time, so we can write this equation:

$$\frac{1}{x-1} + \frac{2}{\sqrt{x^2-1}} + \frac{1}{x+1} = 1$$

This can be expressed as the quartic equation: $x^4 - 4x^3 - 2x^2 + 4x + 5 = 0$. Only one positive real root is not extraneous: 4.18112+. We multiply this by 50 to get the desired answer: 209.056+ feet.

Theodore W. Gibson, of the University of Virginia, found that the first form of the above equation can be written as follows, simply by taking the square root of each side:

$$\frac{1}{\sqrt{x-1}} + \frac{1}{\sqrt{x+1}} = 1$$

which is remarkably similar to the equation for the first version of the problem.

Many readers sent analyses of variations of this problem: a square formation marching in a direction parallel to the square's diagonal, formations of regular polygons with more than four sides, circular formations, rotating formations, and so on. Thomas J. Meehan and David Salsburg each pointed out that the problem is the same as that of a destroyer making a square search pattern around a moving ship, and showed how easily it could be solved by vector diagrams on what the Navy calls a "maneuvering board."

36. The assumption that the "lady" is Jean Brown, the stenographer, quickly leads to a contradiction. Her opening remark brings forth a reply from the person with black hair, therefore Brown's hair cannot be black. It also cannot be brown, for then it would match her name. Therefore it must be white. This leaves brown for the color of Professor Black's hair and black for Professor White. But a statement by the person with black hair prompts an exclamation from White, so they cannot be the same person.

It is necessary to assume, therefore, that Jean Brown is a man. Professor White's hair can't be white (for then it would match his or her name), nor can it be black because he (or she) replies to the black-haired person. Therefore it must be brown. If the lady's hair isn't brown, then Professor White is not a lady. Brown is a man, so Professor Black must be the lady. Her hair can't be black or brown, so she must be a platinum blonde.

37. Since the wind boosts the plane's speed from A to B and retards it from B to A, one is tempted to suppose that these forces balance each other so that total travel time for the combined flights will remain the same. This is not the case, because the time during which the plane's speed is boosted is shorter than the time during which it is retarded, so the over-all effect is one of retardation. The total travel time in a wind of constant speed and direction, regardless of the speed or direction, is always greater than if there were no wind.

38. Let x be the number of hamsters originally purchased and also the number of parakeets. Let y be the number of hamsters among the seven unsold pets. The number of parakeets among the seven will then be $7 - y$. The number of hamsters sold (at a price of $2.20 each, which is a markup of 10 per cent over cost) will be $x - y$, and the number of parakeets sold (at $1.10 each) will be $x - 7 + y$.

The cost of the pets is therefore $2x$ dollars for the hamsters and x dollars for the parakeets—a total of $3x$ dollars. The hamsters that were sold brought $2.2\,(x - y)$ dollars and the parakeets sold brought $1.1\,(x - 7 + y)$ dollars—a total of $3.3x - 1.1y - 7.7$ dollars.

We are told that these two totals are equal, so we equate them and simplify to obtain the following Diophantine equation with two integral unknowns:

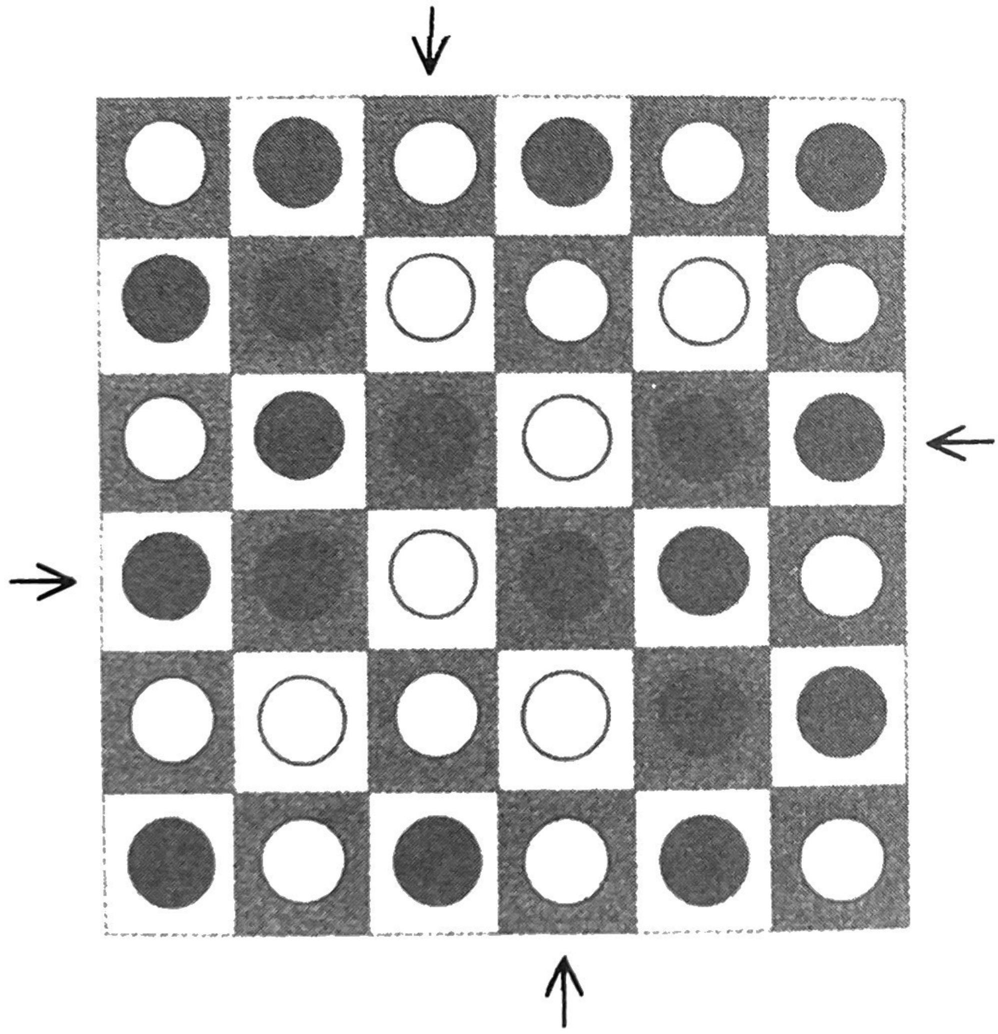
$$3x = 11y + 77$$

Since x and y are positive integers and y is not more than 7, it is a simple matter to try each of the eight possible values (including zero) for y to determine which of them makes x also integral. There are only two such values: 5 and 2. Each would lead to a solution of the problem were it not for the fact that the parakeets were bought in pairs. This eliminates 2 as a value for y because it would give x (the number of parakeets purchased) the odd value of 33. We conclude therefore that y is 5.

A complete picture can now be drawn. The shop owner bought 44 hamsters and 22 pairs of parakeets, paying altogether $132 for them. He sold 39 hamsters and 21 pairs of parakeets for a total of $132. There remained five hamsters worth $11 retail and two parakeets worth $2.20 retail—a combined value of $13.20, which is the answer to the problem.

39. The illustration shows the finish of a drawn game of Hip. This beautiful, hard-to-find solution was first discovered by C. M. McLaury, a mathematics student at the University of Oklahoma to whom I had communicated the problem by way of Richard Andree, one of his professors.

Two readers (William R. Jordan, Scotia, New York, and Donald L. Vanderpool, Towanda, Pennsylvania) were able to show, by an exhaustive enumeration of possibilities, that the solution is unique except for slight variations in the four border cells indicated by arrows. Each cell may be either color, provided all four are not the same color, but since each player is limited in the game to eighteen pieces, two of these cells must be one color, two the other color. They are arranged here so that no matter how the square is turned, the pattern is the same when inverted.



The order-6 board is the largest on which a draw is possible. This was proved in 1960 by Robert I. Jewett, then a graduate student at the University of Oregon. He was able to show that a draw is impossible on the order-7, and since all higher squares contain a seven-by-seven subsquare, draws are clearly impossible on them also.

As a playable game, Hip on an order-6 board is strictly for the squares. David H. Templeton, professor of chemistry at the University of California's Lawrence Radiation Laboratory in Berkeley, pointed out that the second player can always force a draw by playing a simple symmetry strategy. He can either make each move so that it matches his opponent's last move by reflection across a parallel bisector of the board, or by a 90-degree rotation about the board's center. (The latter strategy could lead to the draw depicted.) An alternate strategy is to play in the corresponding opposite cell on a line from the opponent's last move and across the center of the board. Second-player draw strategies were also sent by Allan W. Dickinson, Richmond Heights, Missouri, and Michael Merritt, a student at Texas A. & M. College. These strategies apply to all even-order fields, and since no draws are possible on such fields higher than 6, the strategy guarantees a win for the second player on all even-order boards of 8 or higher. Even on the order-6, a reflection strategy across a parallel bisector is sure to win, because the unique draw pattern does not have that type of symmetry.

Symmetry play fails on odd-order fields because of the central cell. Since nothing is known about strategies on odd-order boards, the order-7 is the best field for actual play. It cannot end in a draw, and no one at present knows whether the first or second player wins if both sides play rationally.

In 1963 Walter W. Massie, a civil engineering student at Worcester Polytechnic Institute, devised a Hip-playing program for the IBM 1620 digital computer, and wrote a term paper about it. The program allows the computer to play first or second on any square field of orders 4 through 10. The computer takes a random cell if it moves first. On other plays, it follows a reflection strategy except when a reflected move forms a square, then it makes random choices until it finds a safe cell.

On all square fields of order n, the number of different squares that can be formed by four cells is $(n^4 - n^2)/12$. The derivation of this formula, as well as a formula for rectangular boards, is given in Harry Langman, *Play Mathematics*, Hafner, 1962, pages 36–37.

As far as I know, no studies have been made of comparable "triangle-free" colorings on triangular lattice fields.

40. The locomotive can switch the positions of cars A and B, and return to its former spot, in 16 operations:

1. Locomotive moves right, hooks to car A.
2. Pulls A to bottom.
3. Pushes A to left, unhooks.

4. Moves right.
5. Makes a clockwise circle through tunnel.
6. Pushes B to left. All three are hooked.
7. Pulls A and B to right.
8. Pushes A and B to top. A is unhooked from B.
9. Pulls B to bottom.
10. Pushes B to left, unhooks.
11. Circles counterclockwise through tunnel.
12. Pushes A to bottom.
13. Moves left, hooks to B.
14. Pulls B to right.
15. Pushes B to top, unhooks.
16. Moves left to original position.

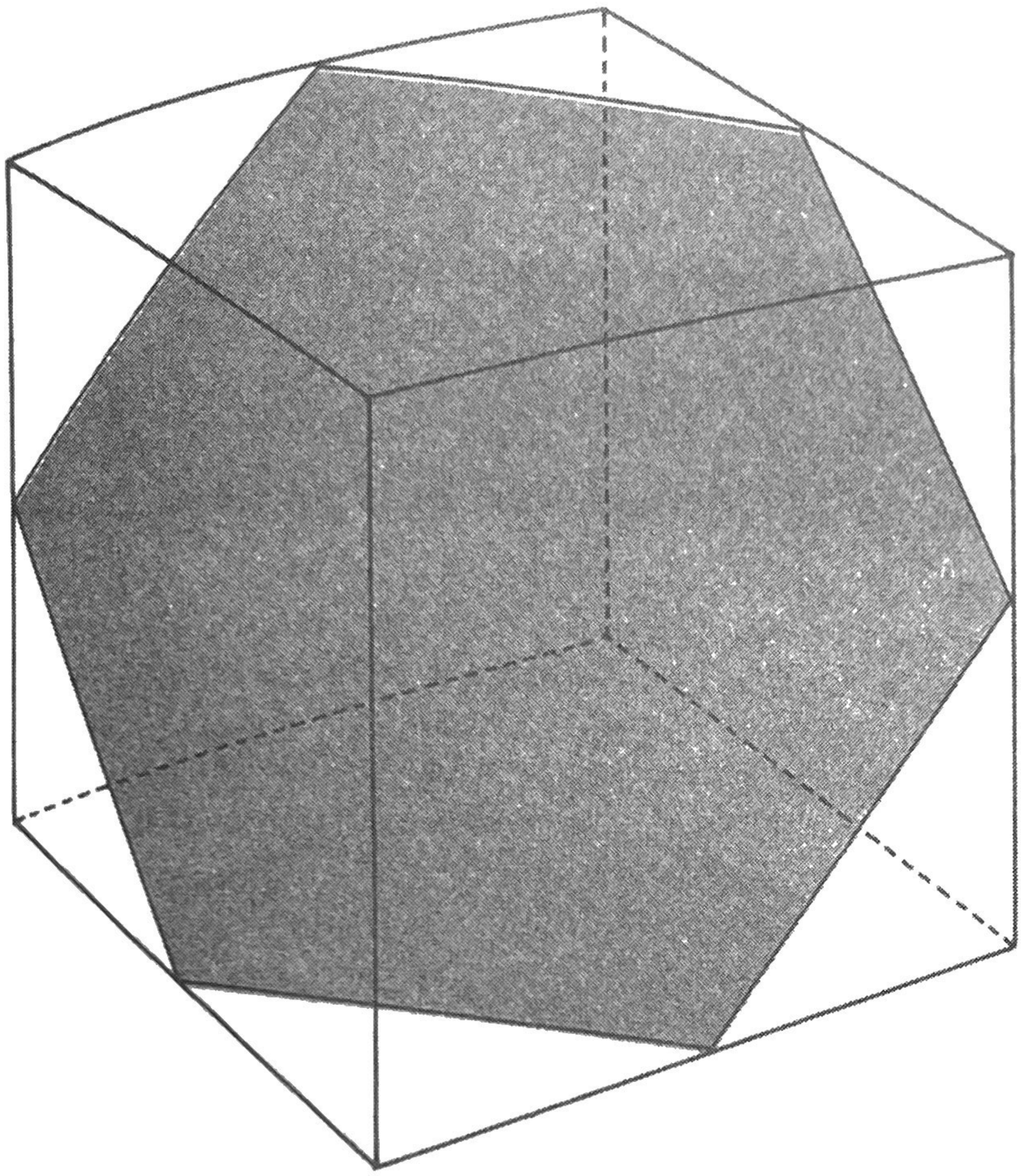
This procedure will do the job even when the locomotive is not permitted to pull with its front end, provided that at the start the locomotive is placed with its back toward the cars.

Howard Grossman, New York City, and Moises V. Gonzalez, Miami, Florida, each pointed out that if the lower siding is eliminated completely, the problem can still be solved, although two additional moves are required, making 18 in all. Can the reader discover how it is done?

Many different train-switching puzzles can be found in the puzzle books by Sam Loyd, Ernest Dudeney and others. Two recent articles deal with this type of puzzle: A. K. Dewdney's Computer Recreations column in *Scientific American* (June 1987) and "Reversing Trains: A Turn of the Century Sorting Problem," by Nancy Amato *et al.*, *Journal of Algorithms,* Vol. 10, 1989, pages 413–428.

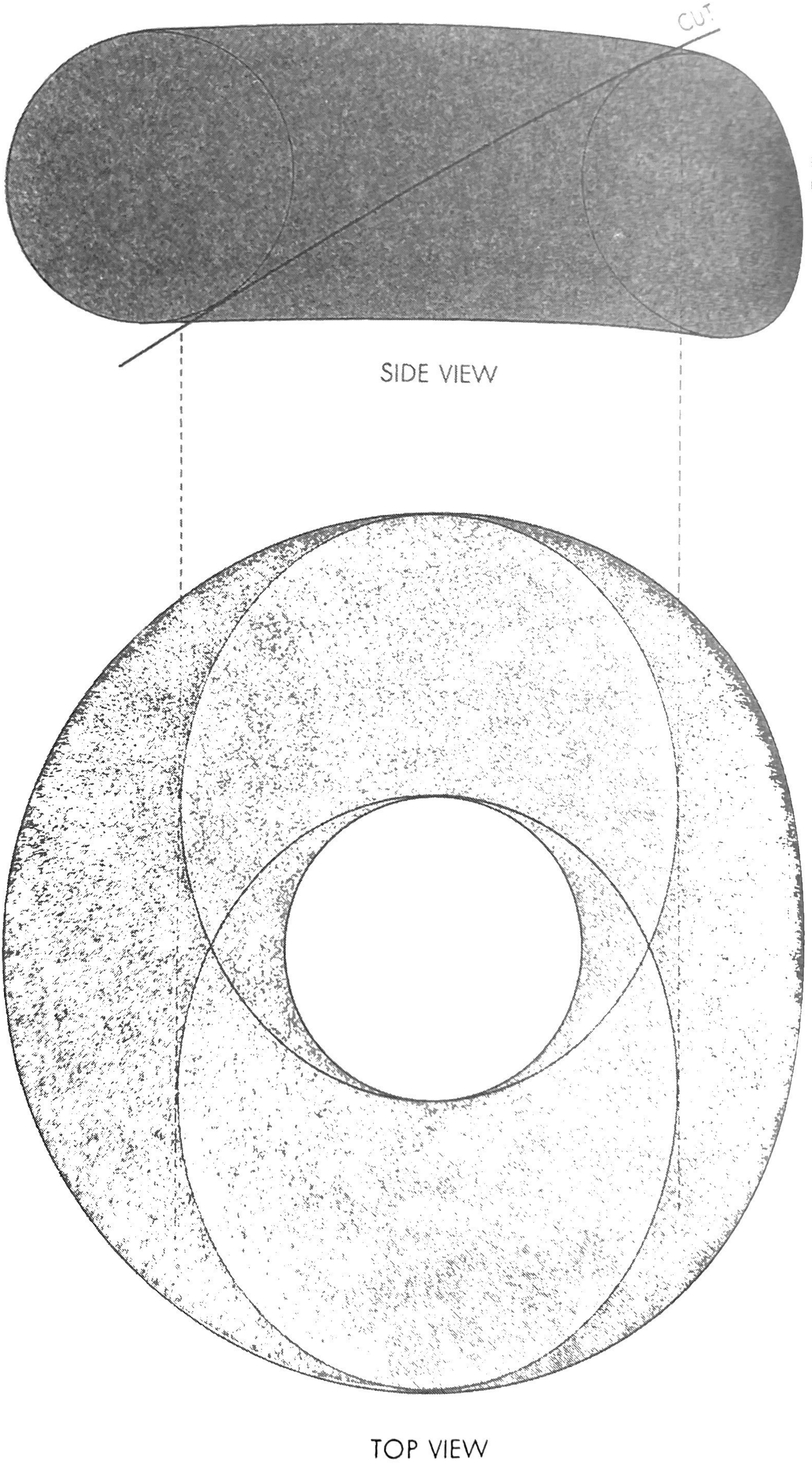
41. The curious thing about the problem of the Flatz beer signs is that it is not necessary to know the car's speed to determine the spacing of the signs. Let x be the number of signs passed in one minute. In an hour the car will pass $60x$ signs. The speed of the car, we are told, is $10x$ miles per hour. In $10x$ miles it will pass $60x$ signs, so in one mile it will pass $60x/10x$, or 6, signs. The signs therefore are 1/6 mile, or 880 feet, apart.

42. A cube, cut in half by a plane that passes through the midpoints of six sides as shown, produces a cross section that is a regular hexagon. If the cube is half an inch on the side, the side of the hexagon is $\sqrt{2}/4$ inch.



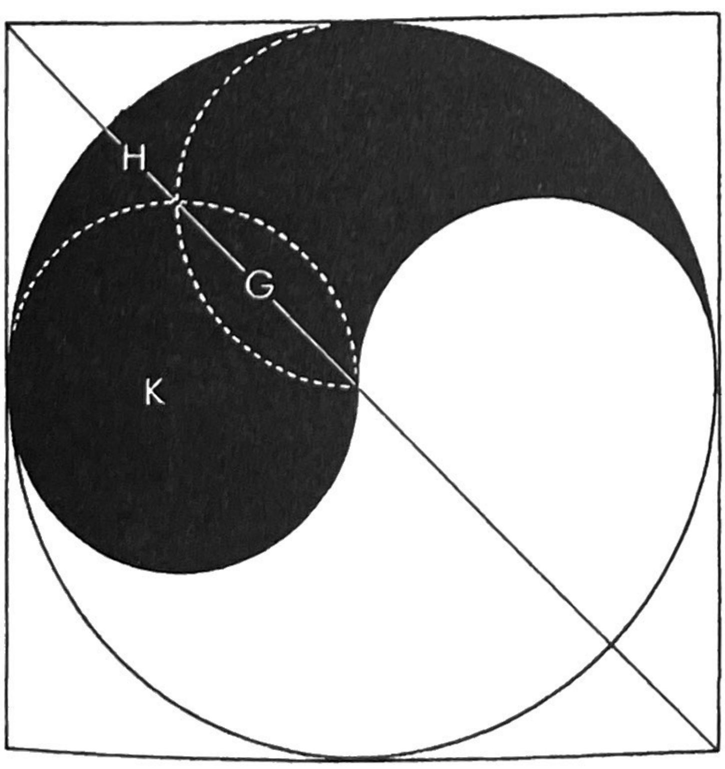
To cut a torus so that the cross section consists of two intersecting circles, the plane must pass through the center and be tangent to the torus above and below, as shown in the second picture (next page). If the torus and hole have diameters of three inches and one inch, each circle of the section will clearly have a diameter of two inches.

This way of slicing, and the two ways described earlier, are the only ways to slice a doughnut so that the cross sections are circular. Everett A. Emerson, in the electronics division of National Cash Register, Hawthorne, California, sent a full algebraic proof that there is no fourth way.


CUT
SIDE VIEW
TOP VIEW

43. The illustration shows how to construct a straight line that bisects both the Yin and the Yang. A simple proof is obtained by drawing the two broken semicircles. Circle K's diameter is half that of the monad; therefore its area is one-fourth that of the monad. Take region G from this circle, add H, and the resulting region is also one-fourth the monad's area. It follows that area G equals area H, and of course half of G must equal half of H. The bisecting line takes half of G away from circle K, but restores the same area (half of H) to the circle, so the black area below the bisecting line must have the same area as circle K. The small circle's area is one-fourth the large circle's area, therefore the Yin is bisected. The same argument applies to the Yang.

The foregoing proof was given by Henry Dudeney in his answer to problem 158, *Amusements in Mathematics* (Thomas Nelson & Sons, 1917; Dover reprint, 1958). After it appeared in *Scientific American,* four readers (A. E. Decae, F. J. Hooven, Charles W. Trigg and B. H. K. Willoughby) sent the following alternative proof, which is much simpler. Draw a horizontal diameter of the small circle K. The semicircle below this line has an area that is clearly 1/8 that of the large circle. Above the diameter is a 45-degree sector of the large circle (bounded by the small circle's horizontal diameter and the diagonal line) which also is obviously 1/8 the area of the large circle. Together, the semicircle and sector have an area of 1/4 that of the large circle, therefore the diagonal line must bisect both Yin and Yang. For ways of bisecting the Yin and Yang with curved lines, the reader is referred to Dudeney's problem, cited above, and Trigg's article, "Bisection of Yin and of Yang," in *Mathematics Magazine,* Vol. 34, No. 2, November–December 1960, pages 107–108.



The Yin-Yang symbol (called the *T'ai-chi-t'u* in China and the *Tomoye* in Japan) is usually drawn with a small spot of Yin inside the Yang and a small spot of Yang inside the Yin. This symbolizes the fact that the great dualities of life are seldom pure; each usually contains a bit of the other. There is an extensive Oriental literature on the symbol. Sam Loyd, who bases several puzzles on the figure (*Sam Loyd's Cyclopedia of Puzzles,* page 26), calls it the Great Monad. The term "monad" is repeated by Dudeney, and also used by Olin D. Wheeler in a booklet entitled *Wonderland,* published in 1901 by the Northern Pacific Railway. Wheeler's first chapter is devoted to a history of the trademark, and is filled with curious information and color reproductions from Oriental sources.

For more on the symbol, see Schuyler Cammann, "The Magic Square of Three in Old Chinese Philosophy and Religion," *History of Religions,* Vol. 1, No. 1, Summer 1961, pages 37–80, my *New Ambidextrous Universe* (W. H. Freeman, 1979), pages 219–220, and George Sarton, *A History of Science,* Vol. 1 (Harvard University Press, 1952; Dover reprint, under the title *Ancient Science through the Golden Age of Greece,* 1993), page 11. Carl Gustav Jung cites some English references on the symbol in his introduction to the book of *I Ching* (1929), and there is a book called *The Chinese Monad: Its History and Meaning,* by Wilhelm von Hohenzollern, the date and publisher of which I do not know.

44. There are probably three blue-eyed Jones sisters and four sisters altogether. If there are n girls, of which b are blue-eyed, the probability that two chosen at random are blue-eyed is:

$$\frac{b(b-1)}{n(n-1)}$$

We are told that this probability is 1/2, so the problem is one of finding integral values for b and n that will give the above expression a value of 1/2. The smallest such values are $n = 4$, $b = 3$. The next highest values are $n = 21$, $b = 15$, but it is extremely unlikely that there would be as many as 21 sisters, so four sisters, three of them blue-eyed, is the best guess.

45. The rose-red city's age is seven billion years. Let x be the city's present age; y, the present age of Time. A billion years ago the city would have been $x - 1$ billion years old and a billion years from now Time's age will be $y + 1$. The data in the problem permit two simple equations:

$$2x = y$$

$$x - 1 = \frac{2}{5}(y + 1)$$

These equations give x, the city's present age, a value of seven billion years; and y, Time's present age, a value of fourteen billion years. The problem presupposes a "Big Bang" theory of the creation of the cosmos.

46. There is space only to suggest the procedure by which it can be shown that Washington High won the high jump event in the track meet involving three schools. Three different positive integers provide points for first, second and third place in each event. The integer for first place must be at least 3. We know there are at least two events in the track meet, and that Lincoln High (which won the shot-put) had a final score of 9, so the integer for first place cannot be more than 8. Can it be 8? No, because then only two events could take place and there is no way that Washington High could build up a total of 22 points. It cannot be 7 because this permits no more than three events, and three are still not sufficient to enable Washington High to reach a score of 22. Slightly more involved arguments eliminate 6, 4 and 3 as the integer for first place. Only 5 remains as a possibility.

If 5 is the value for first place, there must be at least five events in the meet. (Fewer events are not sufficient to give Washington a total of 22, and more than five would raise Lincoln's total to more than 9.) Lincoln scored 5 for the shot-put, so its four other scores must be 1. Washington can now reach 22 in only two ways: 4, 5, 5, 5, 3 or 2, 5, 5, 5, 5. The first is eliminated because it gives Roosevelt a score of 17, and we know that this score is 9. The remaining possibility gives Roosevelt a correct final tally, so we have the unique reconstruction of the scoring shown in the table.

EVENTS	1	2	3	4	5	SCORE
WASHINGTON	2	5	5	5	5	22
LINCOLN	5	1	1	1	1	9
ROOSEVELT	1	2	2	2	2	9

Washington High won all events except the shot-put, consequently it must have won the high jump.

Many readers sent shorter solutions than the one just given. Two readers (Mrs. Erlys Jedlicka, Saratoga, California, and Albert Zoch, a student at Illinois Institute of Technology) noticed that there was a short cut to the solution based on the assumption that the problem had a unique answer. Mrs. Jedlicka put it this way:

> *Dear Mr. Gardner:*
>
> *Did you know this problem can be solved without any calculation whatever? The necessary clue is in the last paragraph. The solution to the integer equations must indicate without ambiguity which school won the high jump. This can only be done if one school has won all the events, not counting the shot-put; otherwise the problem could not be solved with the information given, even after calculating the scoring and number of events. Since the school that won the shot-put was not the over-all winner, it is obvious that the over-all winner won the remaining events. Hence without calculation it can be said that Washington High won the high jump.*

47. It is not possible for the termite to pass once through the 26 outside cubes and end its journey in the center one. This is easily demonstrated by imagining that the cubes alternate in color like the cells of a three-dimensional checkerboard, or the sodium and chlorine atoms in the cubical crystal lattice of ordinary salt. The large cube will then consist of 13 cubes of one color and 14 of the other color. The termite's path is always through cubes that alternate in color along the way; therefore if the path is to include all 27 cubes, it must begin and end with a cube belonging to the set of 14. The central cube, however, belongs to the 13 set; hence the desired path is impossible.

The problem can be generalized as follows: A cube of even order (an even number of cells on the side) has the same number of cells of one color as it has cells of the other color. There is no central cube, but complete paths may start on any cell and end on any cell of opposite color. A cube of odd order has one more cell of one color than the other, so a complete path must begin and end on the color that is used for the larger set. In odd-order cubes of orders 3, 7, 11, 15, 19 . . . the central cell belongs to the smaller set, so it cannot be the end of any complete path. In odd-order cubes of 1, 5, 9, 13, 17 . . . the central cell belongs to the larger set, so it can be the end of any path that starts on a cell of the same color. No closed path, going through every unit cube, is possible on any odd-order cube because of the extra cube of one color.

Many two-dimensional puzzles can be solved quickly by similar "parity checks." For example, it is not possible for a rook to start at one corner of a chessboard and follow a path that carries it once through every square and ends on the square at the diagonally opposite corner.

48. The dime and penny puzzle can be solved in four moves as follows. Coins are numbered from left to right.

1. Move 3, 4 to the right of 5 but separated from 5 by a gap equal to the width of two coins.
2. Move 1, 2 to the right of 3, 4, with coins 4 and 1 touching.
3. Move 4, 1 to the gap between 5 and 3.
4. Move 5, 4 to the gap between 3 and 2.

49. I originally published a solution showing how the three slices of bread could be toasted in two minutes. Five readers surprised me by cutting the time to 111 seconds. I had overlooked the possibility of partially toasting one side of a slice, removing it, then returning it later to complete the toasting.

Seconds	*Operation*
1–3	Put in slice A.
3–6	Put in B.
6–18	A completes 15 seconds of toasting on one side.
18–21	Remove A.
21–23	Put in C.
23–36	B completes toasting on one side.
36–39	Remove B.
39–42	Put in A, turned.
42–54	Butter B.
54–57	Remove C.
57–60	Put in B.
60–72	Butter C.
72–75	Remove A.
75–78	Put in C.
78–90	Butter A.
90–93	Remove B.
93–96	Put in A, turned to complete the toasting on its partially toasted side.
96–108	A completes its toasting.
108–111	Remove C.

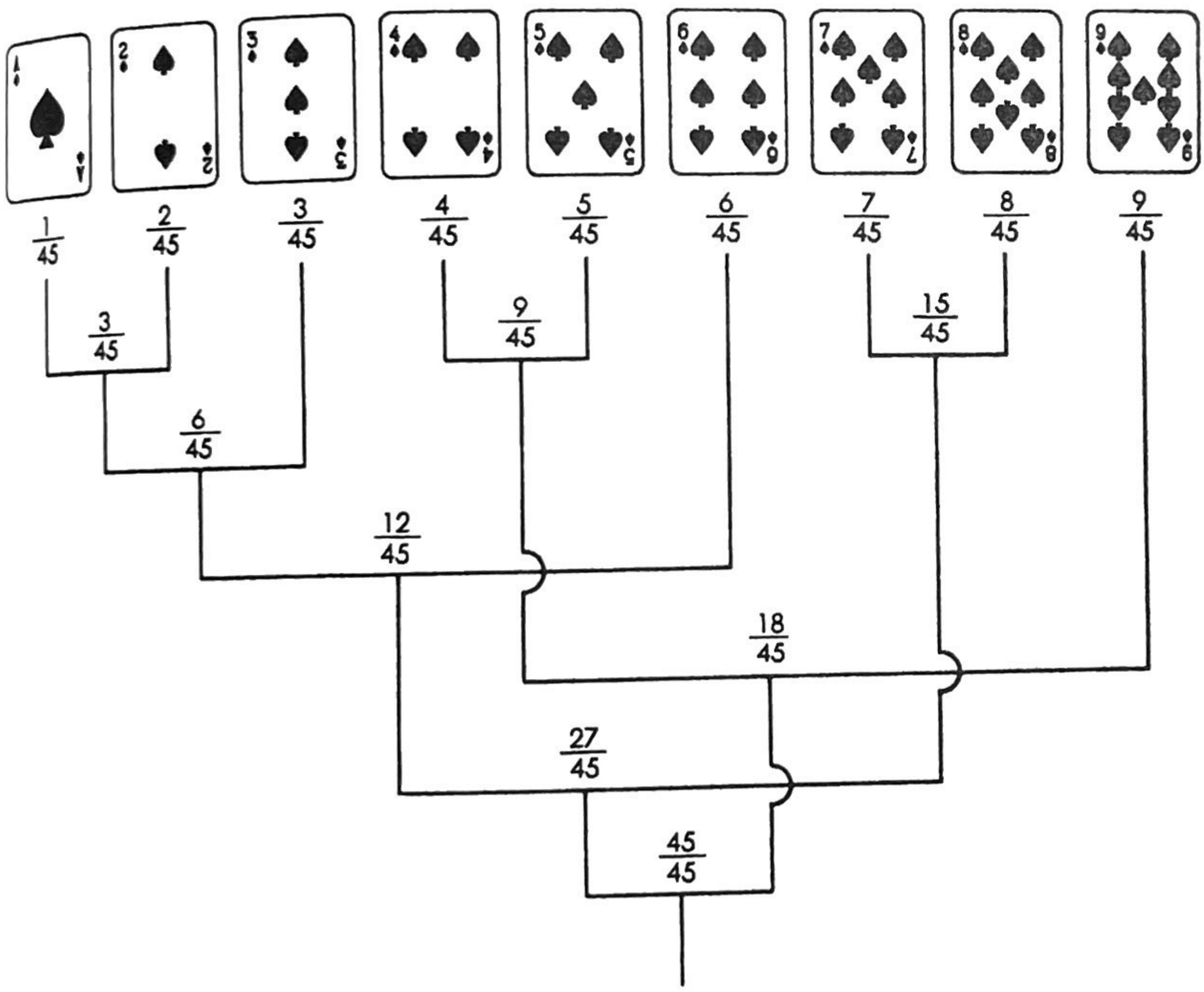
All slices are now toasted and buttered, but slice A is still in the toaster. Even if A must be removed to complete the entire operation, the time is only 114 seconds.

50. A man goes up a mountain one day, down it another day. Is there a spot along the path that he occupies at the same time of day on both trips? This problem was called to my attention by psychologist Ray Hyman, of the University of Oregon, who in turn found it in a monograph entitled "On Problem-Solving," by the German Gestalt psychologist Karl Duncker. Duncker writes of being unable to solve it and of observing with satisfaction that others to whom he put the problem had the same difficulty. There are several ways to go about it, he continues, "but probably none is . . . more drastically evident than the following. Let ascent and descent be divided between *two* persons on the same day. They must *meet.* Ergo. . . . With this, from an unclear dim condition not easily surveyable, the situation has suddenly been brought into full daylight."

51. Immanuel Kant calculated the exact time of his arrival home as follows. He had wound his clock before leaving, so a glance at its face told him the amount of time that had elapsed during his absence. From this he subtracted the length of time spent with Schmidt (having checked Schmidt's hallway clock when he arrived and again when he left). This gave him the total time spent in walking. Since he returned along the same route, at the same speed, he halved the total walking time to obtain the length of time it took him to walk home. This added to the time of his departure from Schmidt's house gave him the time of his arrival home.

52. The first step is to list in order the probability values for the nine cards: 1/45, 2/45, 3/45. . . . The two lowest values are combined to form a new element: 1/45 plus 2/45 equals 3/45. In other words, the probability that the chosen card is either an ace or deuce is 3/45. There are now eight elements: the ace-deuce set, the three, the four, and so on up to nine. Again the two lowest probabilities are combined: the ace-deuce value of 3/45 and the 3/45 probability that the card is a three. This new element, consisting of aces, deuces and threes, has a probability value of 6/45. This is greater than the values for either the fours or fives, so when the two lowest values are combined again, we must pair the fours and fives to obtain an element with the value of 9/45. This procedure of pairing the lowest elements is continued until only one element remains. It will have the probability value of 45/45, or 1.

This chart shows how the elements are combined. The strategy for minimizing the number of questions is to take these pairings in reverse order. Thus the first question could be: Is the card in the set of fours, fives and nines? If not, you know it is in the other set so you ask next: Is it a seven or eight? And so on until the card is guessed.



Note that if the card should be an ace or deuce it will take five questions to pinpoint it. A binary strategy, of simply dividing the elements as nearly as possible into halves for each question, will ensure that no more than four questions need be asked, and you might even guess the card in three. Nevertheless, the previously described procedure will give a slightly lower expected minimum number of questions in the long run; in fact, the lowest possible. In this case, the minimum number is three.

The minimum is computed as follows: Five questions are needed if the card is an ace. Five are also needed if the card is a deuce, but there are two deuces, making ten questions in all. Similarly, the three threes call for three times four, or 12, questions. The total number of questions for all 45 cards is 135, or an average of three questions per card.

This strategy was first discovered by David A. Huffman, an electrical engineer at M.I.T., while he was a graduate student there. It is explained in his paper "A Method for the Construction of Minimum-Redundancy Codes," *Proceedings of the Institute of Radio Engineers,* Vol. 40, pages 1098–1101, September 1952. It was later rediscovered by Seth Zimmerman, who described it in his article on "An Optimal Search Procedure," *American Mathematical Monthly,* Vol. 66, pages 690–693, October 1959. A good nontechnical exposition of the procedure will be found in John R. Pierce, *Symbols, Signals and Noise* (Harper & Brothers, 1961), beginning on page 94.

The eminent mathematician Stanislaw Ulam, in his biography *Adventures of a Mathematician* (Scribner's, 1976, page 281), suggested adding the following rule to the Twenty Questions game. The person who answers is permitted one lie. What is the minimum number of questions required to determine a number between 1 and one million? What if he lies *twice?*

The general case is far from solved. If there are no lies, the answer is of course 20. If just one lies, 25 questions suffice. This was proved by Andrzej Pelc, in "Solution of Ulam's Problem on Searching with a Lie," in *Journal of Combinatorial Theory* (Series A), Vol. 44, pages 129–140, January 1987. The author also gives an algorithm for finding the minimum number of needed questions for identifying any number between 1 and n. A different proof of the 25 minimum is given by Ivan Niven in "Coding Theory Applied to a Problem of Ulam," *Mathematics Magazine,* Vol. 61, pages 275–281, December 1988.

When two lies are allowed, the answer of 29 questions was established by Jurek Czyzowicz, Andrzej Pelc and Daniel Mundici in the *Journal of Combinatorial Theory* (Series A), Vol. 49, pages 384–388, November 1988. In the same journal (Vol. 52, pages 62–76, September 1989), the same authors solved the more general case of two lies and any number between 1 and 2^n. Wojciech Guziki, *ibid.,* Vol. 54, pages 1–19, 1990, completely disposed of the two-lie case for any number between 1 and n.

How about *three* lies? This has been answered only for numbers between 1 and one million. The solution is given by Alberto Negro and Matteo Sereno, in the same journal, Vol. 59, 1992. It is 33 questions, and that's no lie.

The four-lie case remains unsolved even for numbers in the 1 to one million range. Of course if one is allowed to lie every time, there is no way to guess the number. Ulam's problem is closely related to error-correcting coding theory. Ian Stewart summarized the latest results in "How to Play Twenty Questions with a Liar," in *New Scientist* (October 17, 1992), and Barry Cipra did the same in "All Theorems Great and Small," *SIAM News* (July 1992, page 28).

53. In the chess problem white can avoid checkmating black only by moving his rook four squares to the west. This checks the black king, but black is now free to capture the checking bishop with his rook.

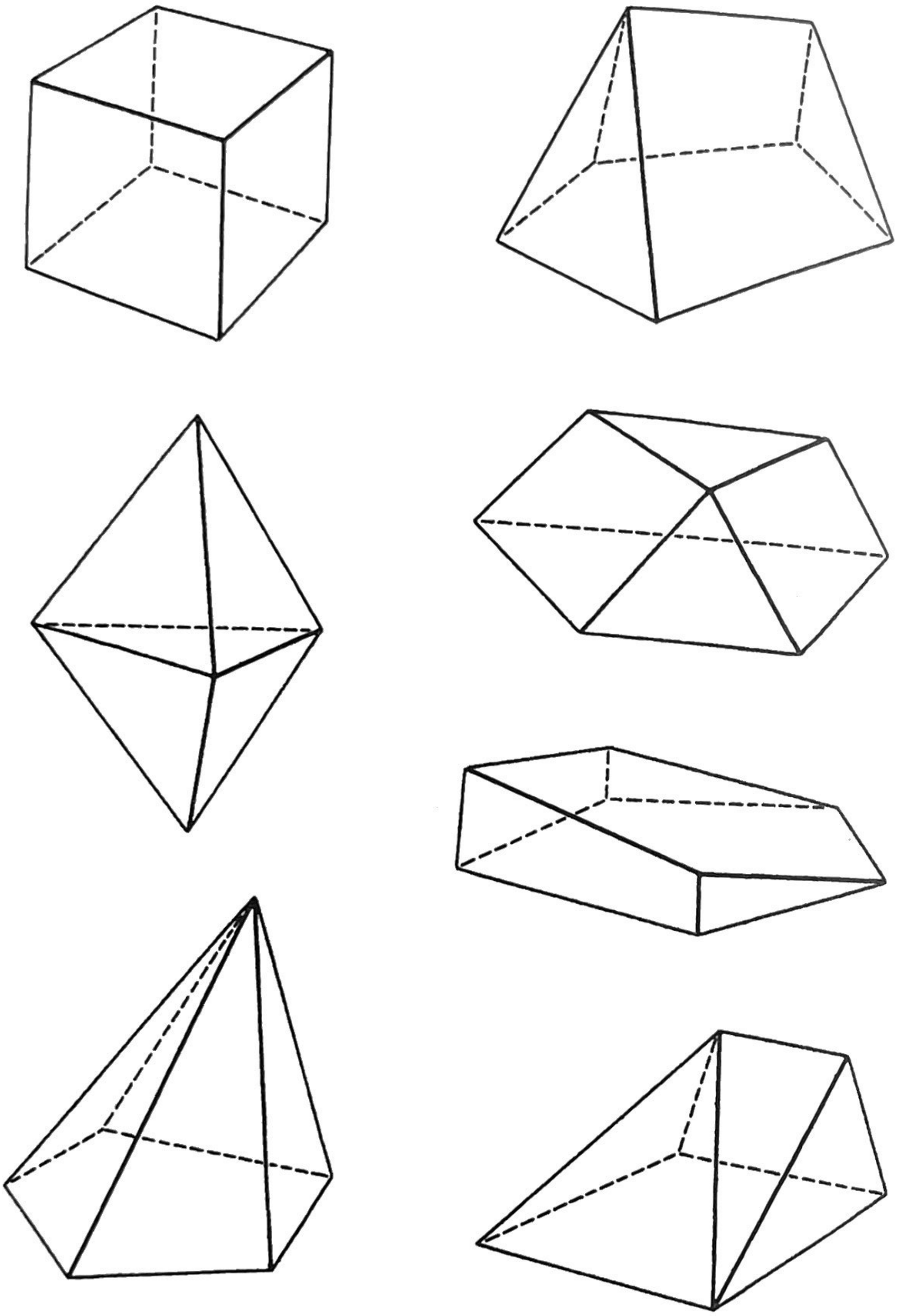
When this problem appeared in *Scientific American,* dozens of readers complained that the position shown is not a possible one because there are two white bishops on the same color squares. They forgot that a pawn on the last row can be exchanged for any piece, not just the queen. Either of the two missing white pawns could have been promoted to a second bishop.

There have been many games by masters in which pawns were promoted to knights. Promotions to bishops are admittedly rare, yet one can imagine situations in which it would be desirable. For instance, to avoid stalemating the opponent. Or white may see that he can use either a new queen or a new bishop in a subtle checkmate. If he calls for a queen, it will be taken by a black rook, in turn captured by a white knight. But if white calls for a bishop, black may be reluctant to trade a rook for bishop and so let the bishop remain.

54. The seven varieties of convex hexahedrons, with topologically distinct skeletons, are shown in the illustration (next page). I know of no simple way to prove that there are no others. An informal proof is given by John McClellan in his article on "The Hexahedra Problem," *Recreational Mathematics Magazine,* No. 4, August 1961, pages 34–40.

There are 34 topologically distinct convex heptahedra, 257 octahedra and 2,606 9-hedra. There are three nonconvex (concave or re-entrant) hexahedra, 26 nonconvex heptahedra and 277 nonconvex octahedra. See the following papers by P. J. Federico: "Enumeration of Polyhedra: The Number of 9-Hedra," *Journal of Combinatorial*

Theory, Vol. 7, September 1969, pages 155–161; "Polyhedra with 4 to 8 Faces," *Geometria Dedicata,* Vol. 3, 1975, pages 469–481; and "The Number of Polyhedra," *Philips Research Reports,* Vol. 30, 1975, pages 220–231.



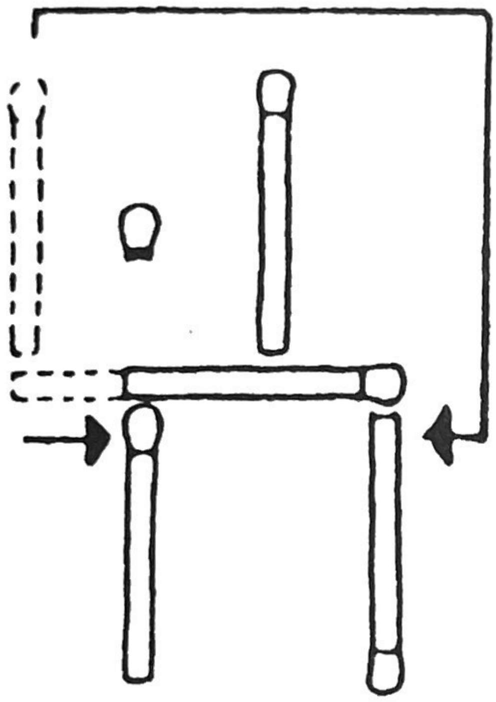
A formula for calculating the number of topologically distinct convex polyhedra, given the number of faces, remains undiscovered.

Paul R. Burnett called my attention to the Old Testament verse Zechariah 3:9. In a modern translation by J. M. Powis Smith it reads:

"For behold the stone which I have set before Joshua; upon a single stone with seven facets I will engrave its inscription."

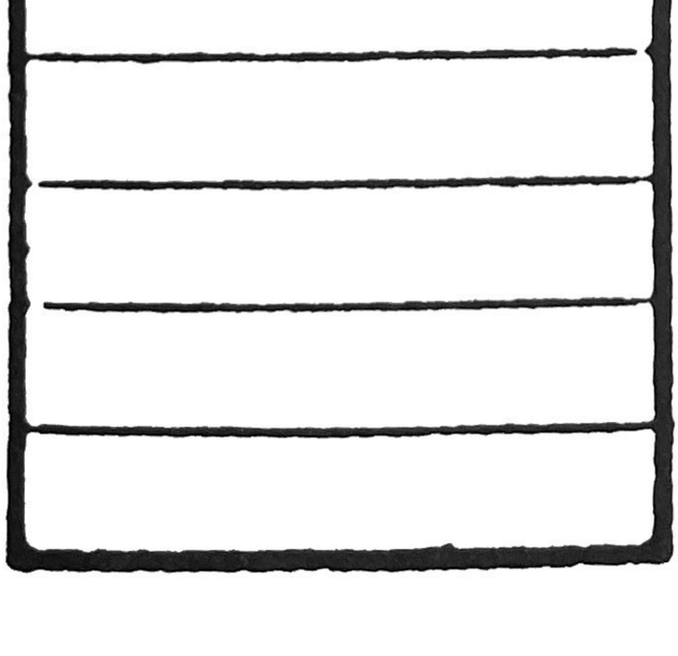
An outline of a formal proof that there are just seven distinct convex hexahedra is given in "Euler's Formula for Polyhedra and Related Topics," by Donald Crowe, in *Excursions into Mathematics*, by Anatole Beck, Michael Bleicher and Donald Crowe (Worth, 1969, pages 29–30).

55.



56. It is only necessary to cut the three links of one piece. They can then be used to join the remaining three pieces into the circular bracelet.

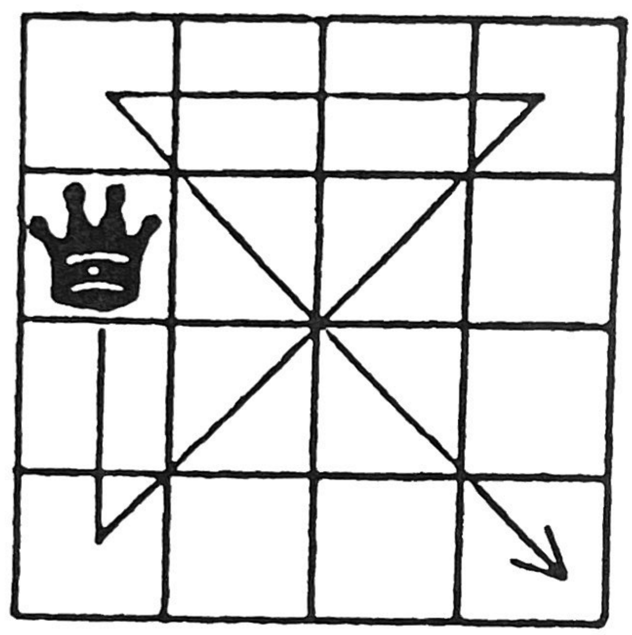
57.



58. Continue the deal by taking cards from the *bottom* of the packet of undealt cards, dealing first to yourself, then counter-clockwise around the table.

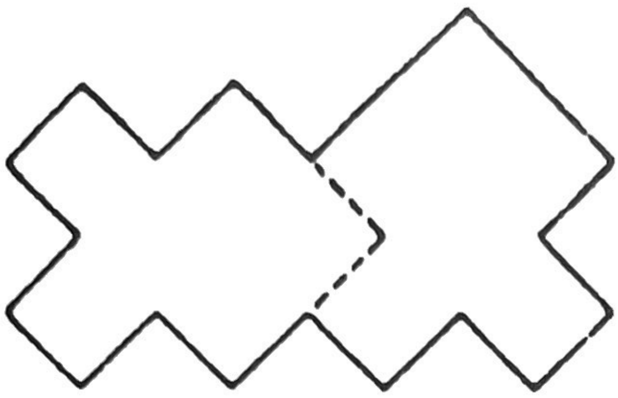
59. Sal wins again. In the first race she ran 100 yards in the time it took Saul to run 90. Therefore, in the second race, after Saul has gone 90 yards, Sal will have gone 100, so she will be alongside him. Both will have 10 more yards to go. Since Sal is the faster runner, she will finish before Saul.

60.



61. The symbols are the numerals 1, 2, 3, 4, 5, 6, 7 shown alongside their mirror reflections. The next symbol, therefore, is the double 8, as shown at the far right, below.

62. The figure is cut into congruent halves like this:



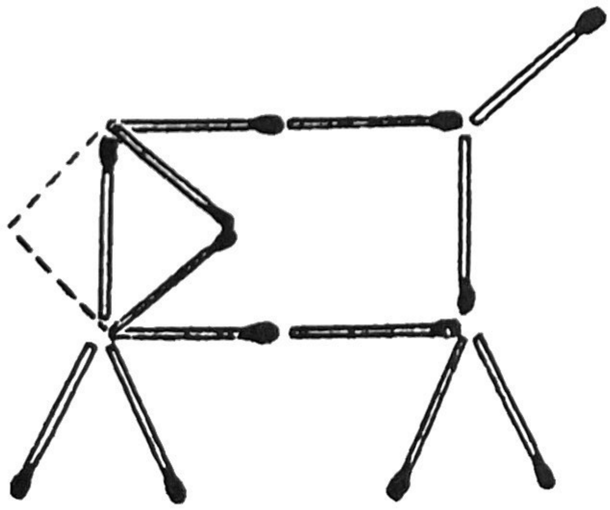
63. Two weighings will do the job. Divide the nine balls into three sets of triplets. Weigh one triplet against another. If a pan goes down you know the heavy ball is among the three on that pan. Pick any two of these balls and weigh one against the other. If one side goes down, you have found the ball. If they balance, the heavy ball must be the one you put aside. In either case, you have found the odd ball in two weighings.

Suppose the two triplets balance on the first weighing. You know then that the heavy ball is in the remaining triplet. As described above, the heavier ball of this triplet is easily identified by weighing any ball of the triplet against any other.

64. Cross out every other letter, starting with N. This eliminates NINE LETTERS, leaving A SINGLE WORD.

After this answer appeared in *Games,* Don Dwyer, Jr., sent to the magazine a second solution. Cross out the nine letters AEILNRSTW every time they appear, to leave the word GOD.

65.



66. As the pictures below show, the folded letter is an upside-down and turned-over F.